# Capacitary estimates of solutions of semilinear parabolic equations

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## 1 Introduction

Let $T \in (0,\infty]$ and $Q_T = \mathbb{R}^N \times (0,T]$ $(N \geq 1)$. If $q > 1$ and $u \in C^2(Q_T)$ is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \tag{1.1}$$

it has been proven by Marcus and Véron [21] that there exists a unique $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, the set of outer-regular positive Borel measures in $\mathbb{R}^N$, such that

$$\lim_{t\to 0} u(.,t) = \nu, \tag{1.2}$$

in the sense of Borel measures. To each such measure $\nu$ is associated a unique couple $(\mathcal{S}_\nu, \mu_\nu)$ (and we write $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$) where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$, *the singular part* of $\nu$, and $\mu_\nu$, *the regular part* is a nonnegative Radon measure on $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$. In this setting, relation (1.2 ) has the following meaning :

$$\begin{array}{lll} (i) & \lim_{t\to 0} \int_{\mathcal{R}_\nu} u(.,t)\zeta dx = \displaystyle\int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) & \lim_{t\to 0} \displaystyle\int_{\mathcal{O}} u(.,t)dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open, } \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{array} \tag{1.3}$$

The measure $\nu$ is by definition the initial trace of $u$ and denoted by $Tr_{\mathbb{R}^N}(u)$. Conversely, in the subcritical range of exponents

$$1 < q < q_c = 1 + N/2,$$

it is proven by the same authors that, for any $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, the Cauchy problem

$$\left\{ \begin{array}{ll} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty, \\ Tr_{\mathbb{R}^N}(u) = \nu, & \end{array} \right. \tag{1.4}$$

admits a unique solution. A key step for proving the uniqueness is the following inequalities

$$t^{-1/(q-1)} f(|x-a|/\sqrt{t}) \leq u(x,t) \leq ((q-1)t)^{-1/(q-1)} \qquad \forall (x,t) \in Q_\infty, \tag{1.5}$$

for any $a \in \mathcal{S}_\nu$, where $f$ is the unique positive solution of

$$\left\{\begin{array}{l} \Delta f + \dfrac{1}{2} y.Df + \dfrac{1}{q-1} f - f^q = 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|y|\to\infty} |y|^{2/(q-1)} f(y) = 0. \end{array}\right. \tag{1.6}$$

The existence, the uniqueness and the asymptotics of $f$ has been proved by Brezis, Peletier and Terman in [5]. The role of the critical exponent $q_c$ was pointed out by Brezis and Friedman [6] who proved that if $q \geq q_c$, *the supercritical range*, any solution of (1.1 ) which vanishes at $t = 0$ for any $x \in \mathbb{R}^N \setminus \{0\}$ must be identically zero. As a consequence, in this range of exponents, Problem (1.4 ) may admit no solution at all. If $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$, the necessary and sufficient conditions for the existence of a maximal solution $u = \overline{u}_\nu$ to Problem (1.4 ) are obtained in [21], and expressed in terms of the the Bessel capacity $C_{2/q,q'}$, (with $q' = q/(q-1)$). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [17]. In the particular case where $\mathcal{S}_\nu = \emptyset$ and $\nu \approx \mu_\nu$, then the necessary and sufficient condition for solvability is that $\mu_\nu$ does not charge Borel subsets with $C_{2/q,q'}$-capacity zero. This result was already proven by Baras and Pierre [4] in the particular case $\nu$ bounded and extended by Marcus and Véron [21] in the general case. We shall denote by $\mathfrak{M}_+^q(\mathbb{R}^N)$ the positive cone of the space $\mathfrak{M}^q(\mathbb{R}^N)$ of Radon measures which does not charge Borel subsets with zero $C_{2/q,q'}$-capacity Notice that $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ is a subset of $\mathfrak{M}_+^q(\mathbb{R}^N)$; here $\mathfrak{M}_+^b(\mathbb{R}^N)$ is the cone of positive bounded Radon mesures in $\mathbb{R}^N$. For such measures, uniqueness always holds and we denote $\overline{u}_\nu = u_\nu$.

The associated stationary equation in a smooth bounded domain $\Omega$ of $\mathbb{R}^N$

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.7}$$

has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a trace for positive solutions, in the class of outer-regular positive borel measures on $\partial\Omega$ is proved by Le Gall [16], in the case $q = N = 2$, by probabilistic methods, and then by Marcus and Véron in [21] in the general case $q > 1$, $N > 1$. The existence of a critical exponent $q_e = (N+1)/(N-1)$ is due to Gmira and Véron. In [8] Dynkin and Kuznetsov introduced the notion of $\sigma$-moderate solution which means that $u$ is a positive solution of (1.7 ) such that there exists an increasing sequence of positive Radon measures on $\partial\Omega$ $\{\mu_n\}$ belonging to $W^{-2/q,q'}(\partial\Omega)$ such that the corresponding solutions $v = v_{\mu_n}$ of

$$\left\{\begin{array}{rl} -\Delta v + v^q = 0 & \text{in } \Omega \\ v = \mu_n & \text{in } \partial\Omega \end{array}\right. \tag{1.8}$$

converges to $u$ locally uniformly in $\Omega$. This class of solutions plays a fundamental role because Dynkin and Kuznetsov proved that a $\sigma$-moderate solution of (1.7 ) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati [27] (if $q = 2$ and then by Dynkin [7] (if $q_e \leq q \leq 2$)), that all the positive solutions of (1.7 ) are $\sigma$-moderate. The key-stone element in their proof is the fact that the maximal solution $\overline{u}_K$ of (1.7 ) the boundary trace of which vanishes outside a compact subset $K \subset \partial\Omega$ is indeed $\sigma$-moderate. This deep result was obtained by a combination of probabilistic

and analytic methods by Mselati in the case $q=2$ and by purely analytic methods by Marcus and Véron [22].

Following Dynkin we can define

**Definition 1.1** *A positive solution $u$ of (1.1 ) is called $\sigma$-moderate if their exists an increasing sequence, say $\{\mu_n\}\subset W^{-2/q,q}(\mathbb{R}^N)\cap\mathfrak{M}_+^b(\mathbb{R}^N)$, such that the corresponding solution $u:=u_{\mu_n}$ of*

$$\left\{\begin{array}{r}\partial_t u-\Delta u+u^q=0 \quad \text{in } Q_\infty\\ u(x,0)=\mu_n \quad \text{in } \mathbb{R}^N,\end{array}\right. \tag{1.9}$$

*converges to $u$ locally uniformly in $Q_\infty$.*

If $F$ is a closed subset of $\mathbb{R}^N$, we denote by $\overline{u}_F$ the maximal solution of (1.1 ) with an initial trace vanishing on $F^c$, and by $\underline{u}_F$ the maximal $\sigma$-moderate solution of (1.1 ) with an initial trace vanishing on $F^c$. Thus $\underline{u}_F$ is defined by

$$\underline{u}_F=\sup\{u_\mu:\mu\in\mathfrak{M}_+^q(\mathbb{R}^N),\mu(F^c)=0\}, \tag{1.10}$$

where $\mathfrak{M}_+^q(\mathbb{R}^N):=W^{-2/q,q}(\mathbb{R}^N)\cap\mathfrak{M}_+^b(\mathbb{R}^N)$. One of the main goal of this article is to prove that $\overline{u}_F$ is $\sigma$-moderate and more precisely,

**Theorem 1.2** *For any $q>1$ and any closed subset $F$ of $\mathbb{R}^N$, $\overline{u}_F=\underline{u}_F$.*

We define below a set function which will play an important role in the sequel.

**Definition 1.3** *Let $F$ be a closed subset of $\mathbb{R}^N$. The $C_{2/q,q'}$-capacitary potential $W_F$ of $F$ is defined by*

$$W_F(x,t)=t^{-1/(q-1)}\sum_{n=0}^{\infty}(n+1)^{N/2-1/(q-1)}e^{-n/4}C_{2/q,q'}\left(\frac{F_n}{\sqrt{(n+1)t}}\right)\quad\forall(x,t)\in Q_\infty, \tag{1.11}$$

*where $F_n=F_n(x,t):=\{y\in F:\sqrt{nt}\leq|x-y|\leq\sqrt{(n+1)t}\}$.*

One of the tool for proving Theorem 1.2 is the following bilateral estimate

**Theorem 1.4** *For any $q\geq q_c$ there exist two positive constants $C_1\geq C_2>0$, depending only on $N$ and $q$ such that for any closed subset $F$ of $\mathbb{R}^N$, there holds*

$$C_2W_F(x,t)\leq\underline{u}_F(x,t)\leq\overline{u}_F(x,t)\leq C_1W_F(x,t)\quad\forall(x,t)\in Q_\infty. \tag{1.12}$$

This *representation* of $\overline{u}_F$, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely what are the blow-up point of $\overline{u}_F$. Introducing an integral expression comparable to $W_F$ we show, in particular, the following results

$$\lim_{\tau\to0}C_{2/q,q'}\left(\frac{F}{\tau}\cap B_1(x)\right)=\gamma\in[0,\infty)\Longrightarrow\lim_{t\to0}t^{-1/(q-1)}\overline{u}_F(x,t)=C\gamma \tag{1.13}$$

for some $C = C(N,q) > 0$, and

$$\limsup_{\tau \to 0} \tau^{2/(q-1)} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty \Longrightarrow \limsup_{t \to 0} \overline{u}_F(x,t) < \infty. \tag{1.14}$$

Our paper is organized as follows. In Section 2 we obtain estimates from above on $\overline{u}_F$. In Section 3 we give estimates from below on $\underline{u}_F$. In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

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## 2 Estimates from above

Some notations : Let $\Omega$ be a domain in $\mathbb{R}^N$ with a compact $C^2$ boundary and $T > 0$. Set $B_r(a)$ the open ball of radius $r > 0$ and center $a$ (and $B_r(0) := B_r$) and

$$Q_T^\Omega := \Omega \times (0,T), \quad \partial_\ell Q_T^\Omega = \partial\Omega \times (0,T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let $\mathbb{H}^\Omega[.]$ (resp. $\mathbb{H}[.]$) denote the heat potential in $\Omega$ with zero lateral boundary data (resp. the heat potential in $\mathbb{R}^N$) with corresponding kernel

$$(x,y,t) \mapsto H^\Omega(x,y,t) \quad (\text{resp.}(x,y,t) \mapsto H(x,y,t) = (4\pi t)^{-N/2} \exp(-|x-y|^2/4t)).$$

We denote by $q_c := 1 + 2/N$, the parabolic critical exponent.

**Theorem 2.1** *Let $q \geq q_c$. Then there exists a positive constant $C_1 = C_1(N,q)$ such that for any closed subset $F$ of $\mathbb{R}^N$ and any $u \in C^2(Q_\infty) \cap C(\overline{Q_\infty} \setminus F)$ satisfying*

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty \\ \lim_{t\to 0} u(x,t) = 0 & \text{locally uniformly in } F^c, \end{array}\right. \tag{2.1}$$

*there holds*

$$u(x,t) \leq C_1 W_F(x,t) \quad \forall (x,t) \in Q_\infty, \tag{2.2}$$

*where $W_F$ is the $(2/q,q')$-capacitary potential of $F$ defined by (1.11 ).*

First we shall consider the case where $F = K$ is compact and

$$K \subset B_r \subset \overline{B}_r, \tag{2.3}$$

and then we shall extend to the general case by a covering argument.

## 2.1 Global $L^q$-estimates

Let $\rho > 0$, we assume (2.3 ) holds and we put

$$\mathcal{T}_{r,\rho}(K) = \{\eta \in C_0^\infty(B_{r+\rho}), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}. \tag{2.4}$$

If $\eta \in \mathcal{T}_{r,\rho}(K)$, we set $\eta^* = 1 - \eta$, $\zeta = \mathbb{H}[\eta^*]^{2q'}$ and

$$R(\eta) = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]|\,. \tag{2.5}$$

We fix $T > 0$ and shall consider the equation on $Q_T$. Throughout this paper $C$ will denote a generic positive constant, depending only on $N$, $q$ and sometimes $T$, the value of which may vary from one ocurrence to another. Except in Lemma 2.12 the only assumption on $q$ is $q > 1$.

**Lemma 2.2** *There exists $C = C(N, q, T) > 0$ such that*

$$\iint_{Q_T} (R(\eta))^{q'}\, dx\, dt \leq C \|\eta\|^{q'}_{W^{2/q,q'}}. \tag{2.6}$$

*Proof.* There holds $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, and

$$\iint_{Q_T} |\partial_t \mathbb{H}[\eta]|^{q'}\, dx\, dt = \int_0^T \left\| t^{1-1/q} \partial_t \mathbb{H}[\eta] \right\|^{q'}_{L^{q'}(\mathbb{R}^N)} \frac{dt}{t} \leq \|\eta\|^{q'}_{\left[W^{2,q'}, L^{q'}\right]_{1/q,q'}} \tag{2.7}$$

where $\left[W^{2,q'}, L^{q'}\right]_{1/q,q'}$ indicates the real interpolation functor of degree $1/q$ between $W^{2,q'}(\mathbb{R}^N)$ and $L^{q'}(\mathbb{R}^N)$ [30]. Similarly, and using the Gagliardo-Nirenberg inequality,

$$\iint_{Q_T} |\nabla (\mathbb{H}[\eta])|^{2q'}\, dx\, dt \leq C \|\eta\|^{q'}_{W^{2/q,q'}} \|\eta\|^{q'}_{L^\infty} = C \|\eta\|^{q'}_{W^{2/q,q'}}. \tag{2.8}$$

Inequality (2.6 ) follows from (2.7 ) and (2.8 ). □

**Lemma 2.3** *There exists $C = C(N, q, T) > 0$ such that*

$$\iint_{Q_T} u^q \zeta dx\, dt + \int_{\mathbb{R}^N} (u\zeta)(x, T) dx \leq C_2 \|\eta\|^{q'}_{W^{2/q,q'}}. \tag{2.9}$$

*Proof.* We recall that there always hold

$$0 \leq u(x, t) \leq \left( \frac{1}{t(q-1)} \right)^{1/(q-1)} \qquad \forall (x, t) \in Q_\infty. \tag{2.10}$$

and (see [6] e.g.)

$$0 \leq u(x, t) \leq \left( \frac{C}{t + (|x| - r)^2} \right)^{1/(q-1)} \qquad \forall (x, t) \in Q_\infty \setminus B_r. \tag{2.11}$$

Since $\eta^*$ vanishes in an open neighborhood $\mathcal{N}_1$, for any open subset $\mathcal{N}_2$ such that $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}}_2 \subset \mathcal{N}_1$ there exist $c_{\mathcal{N}_2} > 0$ and $C_{\mathcal{N}_2} > 0$ such that

$$\mathbb{H}[\eta^*](x,t) \leq C_{\mathcal{N}_2} \exp(-c_{\mathcal{N}_2} t), \quad \forall (x,t) \in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t \to 0} \int_{\mathbb{R}^N} (u\zeta)(x,t)dx = 0,$$

and $\zeta$ is an admissible test function, and one has

$$\iint_{Q_T} u^q \zeta dx\, dt + \int_{\mathbb{R}^N} (u\zeta)(x,T)dx = \iint_{Q_T} u(\partial_t \zeta + \Delta\zeta) dx\, dt. \tag{2.12}$$

Notice that the three terms on the left-hand side are nonnegative. Put $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$, then

$$\begin{aligned} \partial_t \zeta + \Delta\zeta &= 2q'\mathbb{H}_{\eta^*}^{2q'-1} (\partial_t \mathbb{H}_{\eta^*} + \Delta\mathbb{H}_{\eta^*}) + 2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2} |\nabla \mathbb{H}_{\eta^*}|^2, \\ &= 2q'\mathbb{H}_{\eta^*}^{2q'-1} (\partial_t \mathbb{H}_{\eta} + \Delta\mathbb{H}_{\eta}) + 2q'(2q'-1)\mathbb{H}_{\eta}^{2q'-2} |\nabla \mathbb{H}_{\eta}|^2, \end{aligned}$$

because $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_\eta$, hence

$$u(\partial_t\zeta + \Delta\zeta) = u\mathbb{H}_{\eta^*}^{2q'/q} \left[ 2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2-2q'/q} |\nabla \mathbb{H}_\eta|^2 - 2q'\mathbb{H}_{\eta^*}^{2q'-1-2q'/q} (\Delta\mathbb{H}_\eta + \partial_t \mathbb{H}_\eta) \right].$$

Since $2q' - 2 - 2q'/q = 0$ and $0 \leq \mathbb{H}_{\eta^*} \leq 1$,

$$\left| \iint_{Q_T} u(\partial_t\zeta + \Delta\zeta) dx\, dt \right| \leq C(q) \left( \iint_{Q_T} u^q \zeta dx\, dt \right)^{1/q} \left( \iint_{Q_T} R^{q'}(\eta) dx\, dt \right)^{1/q'},$$

where

$$R(\eta) = |\nabla \mathbb{H}_\eta|^2 + |\Delta \mathbb{H}_\eta + \partial_t \mathbb{H}_\eta|.$$

Using Lemma 2.2 one obtains (2.9 ). $\square$

**Proposition 2.4** *Let $r > 0$, $\rho > 0$, $T \geq (r+\rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x,t) : |x|^2 + t \leq (r+\rho)^2\}$$

*and $Q_{r+\rho,T} = Q_T \setminus \mathcal{E}_{r+\rho}$. There exists $C = C(N,q,T) > 0$ such that*

$$\iint_{Q_{r+\rho,T}} u^q dx\, dt + \int_{\mathbb{R}^N} u(x,T)dx \leq C C_{2/q,q'}^{B_{r+\rho}}(K). \tag{2.13}$$

*Proof.* Because $K \subset B_r$ and $\eta^* \equiv 1$ outside $B_{r+\rho}$ and takes value between 0 and 1,

$$\begin{aligned} \mathbb{H}[\eta^*](x,t) \geq \mathbb{H}[1 - \chi_{B_{r+\rho}}](x,t) &= \left(\frac{1}{4\pi t}\right)^{N/2} \int_{|y| \geq r+\rho} \exp(-|x-y|^2/4t) dy, \\ &= 1 - \left(\frac{1}{4\pi t}\right)^{N/2} \int_{|y| \leq r+\rho} \exp(-|x-y|^2/4t) dy. \end{aligned}$$

For $(x,t) \in \mathcal{E}_{r+\rho}$, put $x = (r+\rho)\xi$, $y = (r+\rho)\upsilon$ and $t = (r+\rho)^2\tau$. Then $(\xi,\tau) \in \mathcal{E}_1$ and

$$\left(\frac{1}{4\pi t}\right)^{N/2} \int_{|y|\leq r+\rho} \exp(-|x-y|^2/4t)dy = \left(\frac{1}{4\pi\tau}\right)^{N/2} \int_{|\upsilon|\leq 1} \exp(-|\xi-\upsilon|^2/4\tau)d\upsilon.$$

We claim that

$$\max\left\{\left(\frac{1}{4\pi\tau}\right)^{N/2} \int_{|\upsilon|\leq 1} \exp(-|\xi-\upsilon|^2/4\tau)d\upsilon : (\xi,\tau) \in \mathcal{E}_1\right\} = \ell, \tag{2.14}$$

and $\ell = \ell(N) \in (0,1]$. We recall that

$$\left(\frac{1}{4\pi\tau}\right)^{N/2} \int_{|\upsilon|\leq 1} \exp(-|\xi-\upsilon|^2/4\tau)d\upsilon < 1 \quad \forall \tau > 0. \tag{2.15}$$

If the maximum is achieved for some $(\bar{\xi},\bar{\tau}) \in \mathcal{E}_1$, it is smaller that 1 and

$$\mathbb{H}[\eta^*](x,t) \geq \mathbb{H}[1-\chi_{B_{r+\rho}}](x,t) \geq 1-\ell > 0, \quad \forall (x,t) \in \mathcal{E}_{r+\rho}. \tag{2.16}$$

Let us assume that the maximum is achieved following a sequence $\{(\xi_n,\tau_n)\}$ with $\tau_n \to 0$ and $|\xi_n| \downarrow 1$. We can assume that $\xi_n \to \bar{\xi}$ with $|\bar{\xi}| = 1$, then

$$\left(\frac{1}{4\pi\tau_n}\right)^{N/2} \int_{|\upsilon|\leq 1} e^{-|\xi_n-\upsilon|^2/4\tau_n} d\upsilon = \left(\frac{1}{4\pi\tau_n}\right)^{N/2} \int_{B_1(\xi_n)} e^{-|\upsilon|^2/4\tau_n} d\upsilon.$$

But $B_1(\xi_n) \cap B_1(-\xi_n) = \emptyset$,

$$\int_{B_1(\xi_n)} e^{-|\upsilon|^2/4\tau_n} d\upsilon + \int_{B_1(-\xi_n)} e^{-|\upsilon|^2/4\tau_n} d\upsilon < \int_{\mathbb{R}^N} e^{-|\upsilon|^2/4\tau_n} d\upsilon$$

and

$$\int_{B_1(\xi_n)} e^{-|\upsilon|^2/4\tau_n} d\upsilon = \int_{B_1(-\xi_n)} e^{-|\upsilon|^2/4\tau_n} d\upsilon.$$

This implies

$$\lim_{n\to\infty} \left(\frac{1}{4\pi\tau_n}\right)^{N/2} \int_{B_1(\xi_n)} e^{-|\upsilon|^2/4\tau_n} d\upsilon \leq 1/2.$$

If the maximum were achieved with a sequence $\{(\xi_n,\tau_n)\}$ with $|\tau_n| \to \infty$, it would also imply (2.16 ), since the integral term in (2.15 ) is always bounded. Therefore (2.15 ) holds. Put $C = (1-\ell)^{-1}$, then

$$\iint_{Q_{r,T}} u^q dx\, dt + \int_{\mathbb{R}^N} u(.,T)dx \leq C \|\eta_n\|^{q'}_{W^{2/q,q'}(\mathbb{R}^N)}. \tag{2.17}$$

If we replace $\eta$ by $\eta_n$, a sequence of functions which satisfies

$$C^{B_{r+\rho}}_{2/q,q'}(K) = \lim_{n\to\infty} \|\eta_n\|^{q'}_{W^{2/q,q'}(\mathbb{R}^N)},$$

we obtain (2.13 ). $\square$

## 2.2 Pointwise estimates

We give first a rough pointwise estimate.

**Lemma 2.5** *There exists a constant $C = C(N,q) > 0$ such that*

$$u(x,(r+2\rho)^2) \leq \frac{CC^{B_{r+\rho}}_{2/q,q'}(K)}{(\rho(r+\rho))^{N/2}}, \quad \forall x \in \mathbb{R}^N. \tag{2.18}$$

*Proof. Step 1* We claim that

$$\int_s^T \int_{\mathbb{R}^N} u^q dx\, dt + \int_{\mathbb{R}^N} u(x,T)dx = \int_{\mathbb{R}^N} u(x,s)dx \quad \forall T > s > 0. \tag{2.19}$$

By the maximum principle $u$ is dominated by the solution $v$ with initial trace the indicatrix function $I_{B_r}$. The function $v$ is the limit, as $k \to \infty$, of the solutions $v_k$ with initial data $k\chi_{B_r}$. Since $v_k \leq k\mathbb{H}[\chi_{B_r}]$, it follows Hence

$$\int_{\mathbb{R}^N} u(.,s)dx \leq CC^{B_{r+\rho}}_{2/q,q'}(K) \quad \forall T > s \geq (r+\rho)^2, \tag{2.20}$$

by Lemma 2.3. Using the fact that

$$u(x,\tau+s) \leq \mathbb{H}[u(.,s)](x,\tau) \leq \left(\frac{1}{4\pi\tau}\right)^{N/2} \int_{\mathbb{R}^N} u(.,s)dx,$$

we obtain (2.18 ) with $s = (r+\rho)^2$ and $\tau = (r+2\rho)^2 - (r+\rho)^2 \approx \rho(r+\rho)$. $\square$

The above estimate does not take into account the fact that $u(x,0) = 0$ if $|x| \geq r$. It is mainly interesting if $|x| \leq r$. In order to derive a sharper estimate which uses the localization of the singularity and not only its $C_{2/q,q'}$-capacity, we need some lateral boundary estimates.

**Lemma 2.6** *Let $\gamma \geq r + 2\rho$ and $c > 0$ and either $N = 1$ or $2$ and $0 \leq t \leq c\gamma^2$ for some $c > 0$, or $N \geq 3$ and $t > 0$. Then there holds*

$$\int_0^t \int_{\partial_\ell B_\gamma} udSd\tau \leq C_5\gamma C^{B_{r+\rho}}_{2/q,q'}(K). \tag{2.21}$$

*where $C > 0$ depends on $N$, $q$ and $c$ if $N = 1$, $2$ or depends only on $N$ and $q$ if $N \geq 3$.*

*Proof.* Let us assume that $N = 1$ or 2. Put $G^\gamma := B^c_\gamma \times (-\infty, 0)$ and $\partial_\ell G^\gamma = \partial_\ell B^c_\gamma \times (-\infty, 0)$. Set

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and let $\psi_\gamma$ be the solution of

$$\begin{aligned} \partial_\tau \psi_\gamma + \Delta\psi_\gamma = 0 \qquad & \text{in } G^\gamma, \\ \psi_\gamma = 0 \qquad & \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(.,0) = h_\gamma \qquad & \text{in } B^c_\gamma. \end{aligned} \tag{2.22}$$

Thus the function

$$\tilde{\psi}(x,\tau)=\psi_\gamma(\gamma x,\gamma^2\tau)$$

satisfies

$$\begin{aligned} \partial_t\tilde{\psi}+\Delta\tilde{\psi}&=0 &&\text{in } G^1\\ \tilde{\psi}&=0 &&\text{on } \partial_\ell G^1\\ \tilde{\psi}(.,0)&=\tilde{h} &&\text{in } B_1^c, \end{aligned}\tag{2.23}$$

and $\tilde{h}(x)=1-|x|^{-1}$. By the maximum principle $0\leq\tilde{\psi}\leq 1$, and by Hopf Lemma

$$-\frac{\partial\tilde{\psi}}{\partial\mathbf{n}}\Big|_{\partial B_1^c\times[-c,0]}\geq\theta>0,\tag{2.24}$$

where $\theta=\theta(N,c)$. Then $0\leq\psi_\gamma\leq 1$ and

$$-\frac{\partial\psi_\gamma}{\partial\mathbf{n}}\Big|_{\partial B_\gamma^c\times[-\gamma^2,0]}\geq\theta/\gamma.\tag{2.25}$$

Multiplying (1.1 ) by $\psi_\gamma(x,\tau-t)=\psi_\gamma^*(x,\tau)$ and integrating on $B_\gamma^c\times(0,t)$ yields to

$$\int_0^t\int_{B_\gamma^c}u^q\psi_r^*dxd\tau+\int_{B_\gamma^c}(uh_\gamma)(x,t)dx-\int_0^t\int_{\partial B_\gamma}\frac{\partial u}{\partial\mathbf{n}}\psi_\gamma^*dSd\tau=-\int_0^t\int_{\partial B_\gamma}\frac{\partial\psi_\gamma^*}{\partial\mathbf{n}}ud\sigma d\tau.\tag{2.26}$$

Since $\psi_\gamma^*$ is bounded from above by 1, (2.21 ) follows from (2.25 ) and Proposition 2.4 (notice that $B_\gamma^c\times(0,t)\subset\mathcal{E}_\gamma^c$), first by taking $t=T=\gamma^2\geq(r+2\rho)^2$, and then for any $t\leq\gamma^2$.

If $N\geq 3$, we proceed as above except that we take

$$h_\gamma(x)=1-\left(\frac{\gamma}{|x|}\right)^{N-2}$$

Then $\psi_\gamma(x,t)=h_\gamma(x)$ and $\theta=N-2$ is independent of the length of the time interval. This leads to the conclusion. □

**Lemma 2.7** *I- Let $M$, $a>0$ and $\eta\in L^\infty(\mathbb{R}^N)$ such that*

$$0\leq\eta(x)\leq Me^{-a|x|^2},\quad a.e.\ in\ \mathbb{R}^N.\tag{2.27}$$

*Then, for any $t>0$,*

$$0\leq\mathbb{H}[\eta](x,t)\leq\frac{M}{(4at+1)^{N/2}}e^{-a|x|^2/(4at+1)},\quad\forall x\in\mathbb{R}^N.\tag{2.28}$$

*II- Let $M$, $a$, $b>0$ and $\eta\in L^\infty(\mathbb{R}^N)$ such that*

$$0\leq\eta(x)\leq Me^{-a(|x|-b)_+^2},\quad a.e.\ in\ \mathbb{R}^N.\tag{2.29}$$

*Then, for any $t>0$,*

$$0\leq\mathbb{H}[\eta](x,t)\leq\frac{Me^{-a(|x|-b)_+^2/(4at+1)}}{(4at+1)^{N/2}},\forall x\in\mathbb{R}^N,\ \forall t>0.\tag{2.30}$$

*Proof.* For the first statement, put $a = 1/4s$. Then

$$0 \leq \eta(x) \leq M(4\pi s)^{N/2} \frac{1}{(4\pi s)^{N/2}} e^{-|x|^2/4s} = C(4\pi s)^{N/2} \mathbb{H}[\delta_0](x, s).$$

By the order property of the heat kernel,

$$0 \leq \mathbb{H}[\eta](x,t) \leq M(4\pi s)^{N/2} \mathbb{H}[\delta_0](x, t+s) = M \left( \frac{s}{t+s} \right)^{N/2} e^{-|x|^2/(4(t+s))},$$

and (2.28 ) follows by replacing $s$ by $1/4a$.

For the second statement, let $\tilde{a} < a$ and $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$. A direct computation gives $R = e^{a\tilde{a}b^2/(a-\tilde{a})}$, and (2.30 ) implies

$$0 \leq \eta(x) \leq M e^{a\tilde{a}b^2/(a-\tilde{a})} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we obtain

$$0 \leq \mathbb{H}[\eta](x,t) \leq \frac{C e^{a\tilde{a}b^2/(a-\tilde{a})}}{(4\tilde{a}t+1)^{N/2}} e^{-\tilde{a}|x|^2/(4\tilde{a}t+1)}, \quad \forall x \in \mathbb{R}^N, \ \forall t > 0. \tag{2.31}$$

Since for any $x \in \mathbb{R}^N$ and $t > 0$,

$$(4\tilde{a}t+1)^{-N/2} e^{-\tilde{a}|x|^2/(4\tilde{a}t+1)} \leq e^{-a\tilde{a}b^2/(a-\tilde{a})} (4at+1)^{-N/2} e^{-a(|x|-b)^2/(4at+1)},$$

(2.30 ) follows from (2.31 ). □

**Lemma 2.8** *There exists a constant $C = C(N,q) > 0$ such that*

$$u(x, (r+2\rho)^2) \leq C \max\left\{ \frac{r+\rho}{(|x|-r-2\rho)^{N+1}}, \frac{|x|-r-2\rho}{(r+\rho)^{N+1}} \right\} e^{-(|x|-(r+2\rho))^2/4(r+2\rho)^2} C_{2/q,q'}^{B_{r+\rho}}(K), \tag{2.32}$$

*for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.*

*Proof.* We recall that the Dirichlet heat kernel $H^{B_1^c}$ in the complement of $B_1$ satisfies, for some $C = C(N) > 0$,

$$H^{B_1^c}(x', y', t', s') \leq C_7 (t'-s')^{-(N+2)/2} (|x'| - 1) \exp(-|x'-y'|^2/4(t'-s')), \tag{2.33}$$

for $t' > s'$. By performing the change of variable $x' \mapsto (r+2\rho)x'$, $t' \mapsto (r+2\rho)^2 t'$, for any $x \in \mathbb{R}^N \setminus B_{r+2\rho}$ and $0 \leq t \leq T$, one obtains

$$u(x,t) \leq C(|x| - r - 2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-|x-y|^2/4(t-s)}}{(t-s)^{1+N/2}} u(y,s) d\sigma(y) ds. \tag{2.34}$$

The right-hand side term in (2.34 ) is smaller than

$$\max\left\{ \frac{C(|x|-r-2\rho)}{(t-s)^{1+N/2}} e^{-(|x|-r-2\rho))^2/4(t-s)} : s \in (0,t) \right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y,s) d\sigma(y) ds.$$

We fix $t=(r+2\rho)^2$ and $|x|\geq r+3\rho$. Since

$$\max\left\{\frac{e^{-(|x|-r-2\rho)^2/4s}}{s^{1+N/2}}:s\in\big(0,(r+2\rho)^2\big)\right\}$$
$$=(|x|-r-2\rho)^{-2-N}\max\left\{\frac{e^{-1/4\sigma}}{\sigma^{1+N/2}}:0<\sigma<\left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\},$$

a direct computation gives

$$\max\left\{\frac{e^{-1/4\sigma}}{\sigma^{1+N/2}}:0<\sigma<\left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\}$$
$$=\begin{cases}(2N+4)^{1+N/2}e^{-(N+2)/2} & \text{if } r+3\rho\leq|x|\leq(r+2\rho)(1+\sqrt{4+2N}),\\ \left(\dfrac{|x|-r-2\rho}{r+2\rho}\right)^{2+N}e^{-((|x|-r-2\rho)/(2r+4\rho))^2} & \text{if } |x|\geq(r+2\rho)(1+\sqrt{4+2N}).\end{cases}$$

Thus there exists a constant $C(N)>0$ such that

$$\max\left\{\frac{e^{-(|x|-r-2\rho)^2/4s}}{s^{1+N/2}}:s\in\big(0,(r+2\rho)^2\big)\right\}\leq C(N)\rho^{-2-N}e^{-(|x|-(r+2\rho))^2/4(r+2\rho)^2}. \tag{2.35}$$

Combining this estimate with (2.21 ) with $\gamma=r+2\rho$ and (2.34 ), one derives (2.32 ). $\square$

**Lemma 2.9** *There exists a constant $C=C(N,q)>0$ such that*

$$0\leq u(x,(r+2\rho)^2)\leq C\max\left\{\frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}},\frac{1}{(r+\rho)^{N-1}\rho}\right\}e^{-(|x|-r-3\rho)^2/4(r+2\rho)^2}C^{B_{r+\rho}}_{2/q,q'}(K), \tag{2.36}$$

*for every $x\in\mathbb{R}^N\setminus B_{r+3\rho}$.*

*Proof.* This is a direct consequence of the inequality

$$(|x|-r-2\rho)e^{-(|x|-(r+2\rho))^2/4(r+2\rho)^2}\leq\frac{C(r+\rho)^2}{\rho}e^{-(|x|-(r+3\rho))^2/4(r+2\rho)^2},\quad\forall x\in B^c_{r+2\rho}, \tag{2.37}$$

and Lemma 2.8. $\square$

**Lemma 2.10** *There exists a constant $C=C(N,q)>0$ such that the following estimate holds*

$$u(x,t)\leq\frac{C\tilde{M}e^{-(|x|-r-3\rho)_+^2/4t}}{t^{N/2}}C^{B_{r+\rho}}_{2/q,q'}(K),\quad\forall x\in\mathbb{R}^N,\ \forall t\geq(r+2\rho)^2, \tag{2.38}$$

*where*

$$\tilde{M}=\tilde{M}(x,r,\rho)=\begin{cases}(1+r/\rho)^{N/2} & \textit{if } |x|<r+3\rho\\ (r+\rho)^{N+3}/\rho(|x|-r-2\rho)^{N+2} & \textit{if } r+3\rho\leq|x|\leq C_N(r+2\rho)\\ 1+r/\rho & \textit{if } |x|\geq C_N(r+2\rho)\end{cases} \tag{2.39}$$

*with $C_N=1+\sqrt{4+2N}$.*

*Proof.* It follows by the maximum principle

$$u(x,t) \le \mathbb{H}[u(.,(r+2\rho)^2)](x,t-(r+2\rho)^2).$$

for $t \ge (r+2\rho)^2$ and $x \in \mathbb{R}^N$. By Lemma 2.5 and Lemma 2.9

$$u(x,(r+2\rho)^2) \le C_{10}\tilde{M}e^{-(|x|-r-3\rho)^2/4(r+2\rho)^2}C^{B_{r+2\rho}}_{2/q,q'}(K),$$

where

$$\tilde{M} = \begin{cases} ((r+\rho)\rho)^{-N/2} & \text{if } |x| < r+3\rho \\ (r+\rho)^3/\rho\,(|x|-r-2\rho))^{N+2} & \text{if } r+3\rho \le |x| \le C_N(r+2\rho) \\ 1/(r+\rho)^{N-1}\rho & \text{if } |x| \ge C_N(r+2\rho) \end{cases}$$

Applying Lemma 2.7 with $a = (2r+4\rho)^{-2}$, $b = r+3\rho$ and $t$ replaced by $t-(r+2\rho)^2$ implies

$$u(x,t) \le C\frac{(r+2\rho)^N\tilde{M}}{t^{N/2}}e^{-(|x|-r-3\rho)^2/4t}C^{B_{r+\rho}}_{2/q,q'}(K), \tag{2.40}$$

for all $x \in B^c_{r+3\rho}$ and $t \ge (r+2\rho)^2$, which is (2.38 ). □

The next estimate gives a precise upper bound for $u$ when $t$ is not bounded from below.

**Lemma 2.11** *Assume that* $0 < t \le (r+2\rho)^2$ *for some* $c > 0$*, then there exists a constant* $C = C(N,q) > 0$ *such that the following estimate holds*

$$u(x,t) \le C(r+\rho)\max\left\{\frac{1}{(|x|-r-2\rho)^{N+1}}, \frac{1}{\rho t^{N/2}}\right\}e^{-(|x|-r-3\rho)^2/4t}C^{B_{r+\rho}}_{2/q,q'}(K), \tag{2.41}$$

*for any* $(x,t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0,(r+2\rho)^2]$.

*Proof.* By using (2.21 ) the following estimate is a straightforward variant of (2.32 ) for any $\gamma \ge r+2\rho$,

$$u(x,t) \le C_8(|x|-r-2\rho)(r+2\rho)\max\left\{\frac{e^{-(|x|-r-2\rho)^2/4s}}{s^{1+N/2}} : 0 < s \le t\right\}C^{B_{r+2\rho}}_{2/q,q'}(K). \tag{2.42}$$

Clearly

$$\max\left\{\frac{e^{-(|x|-r-2\rho)^2/4s}}{s^{1+N/2}} : 0 < s \le t\right\}$$
$$= \begin{cases} (2N+4)^{1+N/2}(|x|-r-2\rho)^{-N-2}e^{-(N+2)/2} & \text{if } 0 < |x| \le r+2\rho+\sqrt{2t(N+2)} \\ \dfrac{e^{-(|x|-r-2\rho)^2/4t}}{t^{1+N/2}} & \text{if } |x| > r+2\rho+\sqrt{2t(N+2)}. \end{cases}$$

By elementary analysis, if $x \in B^c_{r+3\rho}$,

$$(|x| - r - 2\rho)e^{-(|x|-r-2\rho)^2/4t} \leq e^{-(|x|-r-3\rho)^2/4t} \begin{cases} \rho e^{-\rho^2/4t} & \text{if } 2t < \rho^2 \\ \dfrac{2t}{\rho} e^{-1+\rho^2/4t} & \text{if } \rho^2 \leq 2t \leq 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\rho^2/4t} \leq \frac{4}{\rho},$$

we derive

$$(|x| - r - 2\rho)e^{-(|x|-r-2\rho)^2/4t} \leq \frac{Ct}{\rho} e^{-(|x|-r-3\rho)^2/4t},$$

from which inequality (2.41 ) follows. □

**Lemma 2.12** *Assume $q \geq q_c$. Then there exists a constant $C$ depending on $N$ and $q$ such that for any $r > 0$ and $\rho > 0$, and any Borel set $E \subset B_r$, there holds*

$$C^{B_{r+\rho}}_{2/q,q'}(E) \leq C r^{N-2/(q-1)} \left(1 + \frac{r}{\rho}\right)^{2/(q-1)} C_{2/q,q'}(E/r), \tag{2.43}$$

*where $C_{2/q,q'}(E) := C^{\mathbb{R}^N}_{2/q,q'}(E)$.*

*Proof.* By the scaling property of Bessel capacities (see [1]), since $q \geq q_c$,

$$C^{B_{r+\rho}}_{2/q,q'}(E) = r^{N-2/(q-1)} C^{B_{1+\rho/r}}_{2/q,q'}(E/r),$$

for any Borel set $E \subset B_r$. It is sufficient to prove (2.43 ) when $E' = E/r \subset B_1$ is a compact set, thus

$$C^{B_{1+r/\rho}}_{2/q,q'}(E') = \inf \left\{ \|\zeta\|^{q'}_{W^{2/q,q'}} : \zeta \in C^2_0(B_{1+r/\rho}), 0 \leq \zeta \leq 1, \, \zeta \equiv 1 \text{ on } E' \right\}.$$

Let $\phi \in C^2(\mathbb{R}^N)$ be a radial cut-off function such that $0 \leq \rho \leq 1$, $\rho = 1$ on $B_1$, $\rho = 0$ on $\mathbb{R}^N \setminus B_{1+\rho/r}$, $|\nabla\phi| \leq Cr\rho^{-1}\chi_{B_{1+\rho/r}\setminus B_1}$ and $\left|D^2\phi\right| \leq Cr^2\rho^{-2}\chi_{B_{1+\rho/r}\setminus B_1}$, where $C$ is independent of $r$ and $\rho$. Let $\zeta \in C^2_0(\mathbb{R}^N)$. Then

$$\nabla(\zeta\phi) = \zeta\nabla\phi + \phi\nabla\zeta \, , \; D^2(\zeta\phi) = \zeta D^2\phi + \phi D^2\zeta + 2\nabla\phi \otimes \nabla\zeta.$$

Thus $\|\zeta\phi\|_{L^{q'}(B_{1+\rho/r})} \leq \|\zeta\|_{L^{q'}(\mathbb{R}^N)}$,

$$\int_{B_{1+\rho/r}} |\nabla(\zeta\phi)|^{q'} dx \leq C \left(1 + \frac{r}{\rho}\right)^{q'} \|\zeta\|^{q'}_{W^{1,q'}}$$

and

$$\int_{B_{r+\rho}} \left|D^2(\zeta\phi)\right|^{q'} dx \leq C \left(1 + \frac{r^2}{\rho^2}\right)^{q'} \|\zeta\|^{q'}_{W^{2,q'}} .$$

Finally

$$\|\zeta\phi\|_{W^{2/q,q'}} \leq C\left(1+\frac{r^2}{\rho^2}\right)\|\zeta\|_{W^{2/q,q'}}.$$

Denote by $\mathcal{T}$ the linear mapping $\zeta \mapsto \zeta\phi$. Because

$$W^{2/q,q'} = \left[W^{2,q'}, L^{q'}\right]_{1/q,q'},$$

(here we use the Lions-Petree real interpolation notations and results from [18]), it follows

$$\|\mathcal{T}\|_{\mathcal{L}(W_0^{2/q,q'}(\mathbb{R}^N),W_0^{2/q,q'}(B_{1+\rho/r}))} \leq C(q)\left(1+\frac{r^2}{\rho^2}\right)^{1/q}$$

Therefore

$$C_{2/q,q'}^{B_{1+\rho/r}}(E') \leq C\left(1+\frac{r^2}{\rho^2}\right)^{1/(q-1)} C_{2/q,q'}(E').$$

Thus we get (2.43 ). $\square$

*Remark.* In the subcritical case $1 < q < q_c$, estimate (2.43 ) becomes

$$C_{2/q,q'}^{B_{r+\rho}}(E) \leq C\max\left\{r^N, \rho^N\right\}\left(1+\rho^{-2/(q-1)}\right). \tag{2.44}$$

By using Lemma 2.11, it is easy to derive from this estimate that for any positive solution $u$ of (2.1 ), the initial trace of which vanishes outside 0, there holds

$$u(x,t) \leq Ct^{-1/(q-1)}\min\left\{1, \left(\frac{|x|}{\sqrt{t}}\right)^{2/(q-1)-N} e^{-|x|^2/4t}\right\} \quad \forall (x,t) \in Q_\infty. \tag{2.45}$$

This upper estimate corresponds to the one obtained in [5]. If $F = \overline{B}_r$, the upper we estimate is less esthetic. However, it is proved in [21] by a barrier method that, if the initial trace of positive solution $u$ of (2.1 ), vanishes outside F, and if $1 < q < 3$, there holds

$$u(x,t) \leq t^{-1/(q-1)} f_1((|x|-r)/\sqrt{t}) \quad \forall (x,t) \in Q_\infty,\ |x| \geq r, \tag{2.46}$$

where $=$ $f_1$ is the positive solution belonging to $C^2([0,\infty))$ of

$$\left\{\begin{array}{l} f'' + \dfrac{y}{2}f' + \dfrac{1}{q-1}f - f^q = 0 \quad \text{in } (0,\infty) \\ f'(0) = 0\,,\ \lim_{y\to\infty}|y|^{2/(q-1)} f(y) = 0. \end{array}\right. \tag{2.47}$$

Notice that the existence of $f_1$ follows from [5] since $q$ is the critical exponent in 1 dim. Furthermore $f_1$ has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)}e^{-y^2/4t}(1+\circ(1))) \quad \text{as } y \to \infty.$$

## 2.3 The upper Wiener test

**Definition 2.13** We define on $\mathbb{R}^N \times \mathbb{R}$ the two *parabolic distances* $\delta_2$ and $\delta_\infty$ by

$$\delta_2[(x,t),(y,s)] := \sqrt{|x-y|^2 + |t-s|}, \tag{2.48}$$

and

$$\delta_\infty[(x,t),(y,s)] := \max\{|x-y|, \sqrt{|t-s|}\}. \tag{2.49}$$

If $K \subset \mathbb{R}^N$ and $i = 2, \infty$,

$$\delta_i[(x,t),K] = \inf\{\delta_i[(x,t),(y,0)] : y \in K\} = \begin{cases} \max\left\{\text{dist}\,(x,K), \sqrt{|t|}\right\} & \text{if } i = \infty, \\ \sqrt{\text{dist}^{\,2}(x,K) + |t|} & \text{if } i = 2. \end{cases}$$

For $\beta > 0$ and $i = 2, \infty$, we denote by $\mathcal{B}^i_\beta(m)$ the parabolic ball of center $m = (x,t)$ and radius $\beta$ in the parabolic distance $\delta_i$.

Let $K$ be any compact subset of $\mathbb{R}^N$ and $\overline{u}_K$ the maximal solution of (1.1 ) which blows up on $K$. The function $\overline{u}_K$ is obtained as the decreasing limit of the $\overline{u}_{K_\epsilon}$ ($\epsilon > 0$) when $\epsilon \to 0$, where

$$K_\epsilon = \{x \in \mathbb{R}^N : \text{dist}\,(x,K) \leq \epsilon\}$$

and $\overline{u}_{K_\epsilon} = \lim_{k\to\infty} u_{k,K_\epsilon} = \overline{u}_K$, where $u_k$ is the solution of the classical problem,

$$\begin{cases} \partial_t u_k - \Delta u_k + u_k^q = 0 & \text{in } Q_T, \\ u_k = 0 & \text{on } \partial_\ell Q_T, \\ u_k(.,0) = k\chi_{K_\epsilon} & \text{in } \mathbb{R}^N. \end{cases} \tag{2.50}$$

If $(x,t) = m \in \mathbb{R}^N \times (0,T]$, we set $d_K = \text{dist}\,(x,K)$, $D_K = \max\{|x-y| : y \in K\}$ and $\lambda = \sqrt{d_K^2 + t} = \delta_2[m,K]$. We define a slicing of $K$, by setting $d_n = d_n(K,t) := \sqrt{nt}$ ($n \in \mathbb{N}$),

$$T_n = \overline{B}_{d_{n+1}}(x) \setminus B_{d_n}(x), \quad \forall n \in \mathbb{N},$$

thus $T_0 = B_{\sqrt{t}}(x)$, and

$$K_n(x) = K \cap T_n(x) \;\text{ for } n \in \mathbb{N} \text{ and } \mathcal{Q}_n(x) = K \cap B_{d_{n+1}}(x).$$

When there is no ambiguity, we shall skip the $x$ variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

**Theorem 2.14** *Assume $q \geq q_c$. Then there exists $C = C(N,q,T) > 0$ such that*

$$\overline{u}_K(x,t) \leq \frac{C}{t^{N/2}} \sum_{n=0}^{a_t} d_{n+1}^{N-2/(q-1)} e^{-n/4} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \quad \forall (x,t) \in Q_T, \tag{2.51}$$

*where $a_t$ is the largest integer $j$ such that $K_j \neq \emptyset$.*

With no loss of generality, we can first assume that $x = 0$. Furthermore, in considering the scaling transfoprmation $u_\ell(y,t) = \ell^{1/(q-1)}u(\sqrt{\ell}y, \ell t)$, with $\ell > 0$, we can assume $t = 1$. Thus the new compact singular set of the initial trace becomes $K/\sqrt{\ell}$, that we shall still denote $K$. We shall also set $a_{_K} = a_{_K,1}$ Since for each $n \in \mathbb{N}$,

$$\frac{1}{2\sqrt{n+1}} \leq d_{n+1} - d_n \leq \frac{1}{\sqrt{n+1}},$$

it is possible to exhibit a collection $\Theta_n$ of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j}\in\Theta_n} B_{1/\sqrt{n+1}}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq 1/2\sqrt{n+1} \text{ and } \#\Theta_n \leq Cn^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{1/\sqrt{n+1}}(a_{n,j})$, there holds

$$K = \bigcup_{0\leq n\leq a_{_K}} \bigcup_{a_{n,j}\in\Theta_n} K_{n,j}.$$

The first intermediate step is related to the quasi-additivity property of capacities.

**Lemma 2.15** *Let $q \geq q_c$. There exists a constant $C = C(N,q)$ such that*

$$\sum_{a_{n,j}\in\Theta_n} C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \leq Cn^{1/(q-1)-N/2} C_{2/q,q'}\left(\sqrt{n}\, K_n\right) \quad \forall n \in \mathbb{N}_*, \tag{2.52}$$

*where $B_{n,j} = B_{2/\sqrt{n+1}}(a_{n,j})$ and $C_{2/q,q'}$ stands for the capacity taken with respect to $\mathbb{R}^N$.*

*Proof.* The following result is proved in [2, Th 3]: if the spheres $B_{\rho^\theta_j}(b_j)$ are disjoint in $\mathbb{R}^N$ and $G$ is an analytic subset of $\bigcup B_{\rho_j}(b_j)$ where the $\rho_j$ are positive and smaller than some $\rho^* > 0$, there holds

$$C_{2/q,q'}(G) \leq \sum_j C_{2/q,q'}(G \cap B_{\rho_j}(b_j)) \leq AC_{2/q,q'}(G), \tag{2.53}$$

where $\theta = 1 - 2/N(q-1)$, for some $A$ depending on $N$, $q$ and $\rho^*$. This property is called *quasi-additivity.* We define for $n \in \mathbb{N}_*$,

$$\tilde{T}_n = \sqrt{n}T_n, \quad \tilde{K}_n = \sqrt{n}K_n \quad \text{and } \tilde{\mathcal{Q}}_n = \sqrt{n}\mathcal{Q}_n.$$

Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$, the $C_{2/q,q'}$ capacities are taken with respect to the balls $B_{2/\sqrt{n+1}}(a_{n,j}) = B_{n,j}$. By Lemma 2.12 with $r = \rho = \sqrt{n+1}$

$$C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \leq Cn^{1/(q-1)-N/2} C_{2/q,q'}(\tilde{K}_{n,j}), \tag{2.54}$$

where $\tilde{K}_{n,j} = \sqrt{n}K_{n,j}$ and $\tilde{B}_{n,j} = \sqrt{n}B_{n,j}$. For a fixed $n > 0$ and each repartition $\Lambda$ of points $\tilde{a}_{n,j} = \sqrt{n}\, a_{n,j}$ such that the balls $B_{2^\theta}(\tilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds in the following sense: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j}\in\Lambda} K_{n,j}\ , \quad \tilde{K}_{n,\Lambda} = \sqrt{n}\, K_{n,\Lambda} = \bigcup_{a_{n,j}\in\Lambda} \tilde{K}_{n,j} \quad \text{and } \tilde{K}_n = \sqrt{n}\, K_n,$$

then

$$\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq AC_{2/q,q'}(\tilde{K}_{n,\Lambda}). \tag{2.55}$$

The maximal cardinal of any such repartition $\Lambda$ is of the order of $Cn^{N-1}$ for some positive constant $C = C(N)$, therefore, the number of repartitions needed for a full covering of the set $\tilde{T}_n$ is of finite order depending upon the dimension. Because $\tilde{K}_n$ is the union of the $\tilde{K}_{n,\Lambda}$,

$$\sum_{\Lambda}\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq C\, C_{2/q,q'}(\tilde{K}_n) \tag{2.56}$$

Combining (2.54 ) and (2.56 ), we obtain (2.52 ). $\square$

*Proof of Theorem 2.14. Step 1.* We first notice that

$$\overline{u}_K \leq \sum_{0\leq n\leq a_K}\ \sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j}}. \tag{2.57}$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, for any $0 < \epsilon' < \epsilon$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\,\epsilon}$. Because a finite sum of positive solutions of (1.1 ) is a super solution,

$$\overline{u}_{K_{\epsilon'}} \leq \sum_{0\leq n\leq a_K}\ \sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j\,\epsilon}}. \tag{2.58}$$

Letting successively $\epsilon'$ and $\epsilon$ go to 0 implies (2.57 ).

*Step 2.* Let $n\in\mathbb{N}$. Since $K_{n,j}\subset B_{1/\sqrt{n+1}}(a_{n,j})$ and $|x-a_{n,j}| = (d_n+d_{n+1})/2 = (\sqrt{n+1}+\sqrt{n})/2$, we can apply the previous lemmas with $r = 1/\sqrt{n+1}$ and $\rho = r$. For $n \geq n_N$ there holds $t = 1 \geq (r+2\rho)^2 = 9/(n+1)$ and $|x-a_{n,j}| = (\sqrt{n+1}-\sqrt{n})/2 \geq (2+C_N)(3/\sqrt{n+1})$ (notice that $n_N \geq 8$). Thus

$$\begin{aligned} u_{K_{n,j}}(0,1) &\leq Ce^{(\sqrt{n}-3/\sqrt{n+1})^2/4} C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \\ &\leq Ce^{3/2}e^{-n/4} C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \\ &\leq Cn^{1/(q-1)-N/2}e^{-n/4} C_{2/q,q'}(\tilde{K}_{n,j}), \end{aligned} \tag{2.59}$$

which implies

$$\sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) \leq Cn^{N/2-1/(q-1)}e^{-n/4} C_{2/q,q'}(\tilde{K}_n)$$

Using the fact that

$$C_{2/q,q'}\left(\tilde{K}_n\right) \approx \left(d_{n+1}\sqrt{n}\right)^{N-2/(q-1)} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right),$$

for any $n\in\mathbb{N}_*$, we derive

$$\sum_{n=n_N}^{a_K}\ \sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) \leq C\sum_{n=n_N}^{a_K} d_{n+1}^{N/2-1/(q-1)} e^{-n/4} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.60}$$

Finally, we apply Lemma 2.5 if $1 \leq n < n_{_N}$ and get

$$\begin{aligned}\sum_{1}^{n_{_N}-1}\sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) &\leq C\sum_{1}^{n_{_N}-1} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)\\ &\leq C'\sum_{1}^{n_{_N}-1} d_{n+1}^{N/2-1/(q-1)}e^{-n/4}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right).\end{aligned} \tag{2.61}$$

For $n = 0$, we proceed similarly, in splitting $K_1$ in a finite number of $K_{1,i}$, depending only on the dimension, such that $\operatorname{diam} K_{1,i} < 1/3$. Combining (2.60 ) and (2.61 ), we derive

$$\overline{u}_K(0,1) \leq C\sum_{n=0}^{a_K} d_{n+1}^{N/2-1/(q-1)}e^{-n/4}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.62}$$

In order to derive the same result for any $t > 0$, we notice that

$$\overline{u}_K(y,t) = t^{-1/(q-1)}\overline{u}_{K\sqrt{t}}(y\sqrt{t},1).$$

Going back to the definition of $d_n = d_n(K,t) = \sqrt{nt} = d_n(K\sqrt{t},1)$, we derive from (2.62 ) and the fact that $a_{_{K,t}} = a_{_{K\sqrt{t},1}}$

$$\overline{u}_K(0,t) \leq Ct^{-1/(q-1)}\sum_{n=0}^{a_K}(n+1)^{N/2-1/(q-1)}e^{-n/4}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{2.63}$$

which can also read as (2.51 ) with $x = 0$, and a space translation leads to the final result. □

*Proof of Theorem 2.1.* Let $m > 0$ and $F_m = F \cap \overline{B}_m$. We denote by $U_{B_m^c}$ the maximal solution of (1.1 ) in $Q_\infty$ the initial trace of which vanishes on $B_m$. Such a solution is actually the unique solution of (2.1 ) which satisfies

$$\lim_{t\to 0} u(x,t) = \infty$$

uniformly on $B^c_{m'}$, for any $m' > m$: this can be checked by noticing that

$$U_{B_m^c\,\ell}(y,t) = \ell^{1/(q-1)}U_{B_m^c}(\sqrt{\ell}y,\ell t) = U_{B^c_{m/\sqrt{\ell}}}(y,t).$$

Furthermore

$$\lim_{m\to\infty} U_{B_m^c}(y,t) = \lim_{m\to\infty} m^{-2/(q-1)}U_{B_1^c}(y/m,t/m^2) = 0$$

uniformly on any compact subset of $\overline{Q}_\infty$. Since $\overline{u}_{F_m} + U_{B_m^c}$ is a super-solution, it is larger that $\overline{u}_F$ and therefore $\overline{u}_{F_m} \uparrow \overline{u}_F$. Because $W_{F_m}(x,t) \leq W_F(x,t)$ and $\overline{u}_{F_m} \leq C_1 W_{F_m}(x,t)$, the result follows. □

Theorem 2.1 admits the following integral expression.

**Theorem 2.16** *Assume $q \geq q_c$. Then there exists a positive constant $C_1^* = C^*(N,q,T)$ such that, for any closed subset $F$ of $\mathbb{R}^N$, there holds*

$$\overline{u}_F(x,t) \leq \frac{C_1^*}{t^{1+N/2}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-s^2/4t} s^{N-2/(q-1)} C_{2/q,q'}\left(\frac{1}{s}F \cap B_1(x)\right) s\,ds, \tag{2.64}$$

*where $a_t = \min\{n : F \subset B_{\sqrt{(n+1)t}}(x)\}$.*

*Proof.* We first use

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \leq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right),$$

and we denote

$$\Phi(s) = C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \quad \forall s > 0. \tag{2.65}$$

*Step 1.* The following inequality holds (see [1] and [24])

$$c_1 \Phi(\alpha s) \leq \Phi(s) \leq c_2 \Phi(\beta s) \quad \forall s > 0,\ \forall 1/2 \leq \alpha \leq 1 \leq \beta \leq 2, \tag{2.66}$$

for some positive constants $c_1$, $c_2$ depending on $N$ and $q$. If $\beta \in [1,2]$,

$$\Phi(\beta s) = C_{2/q,q'}\left(\frac{1}{\beta}\left(\frac{F}{s} \cap B_\beta\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\beta\right) \geq c_1 \Phi(s).$$

If $\alpha \in [1/2, 1]$,

$$\Phi(\alpha s) = C_{2/q,q'}\left(\frac{1}{\alpha}\left(\frac{F}{s} \cap B_\alpha\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\alpha\right) \leq c_2 \Phi(s).$$

*Step 2.* By (2.66 )

$$C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \leq c_2 C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \quad \forall\ s \in [d_{n+1}, d_{n+2}],$$

and $n \leq a_t$. Then

$$\begin{aligned} c_2 \int_{d_{n+1}}^{d_{n+2}} & s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\,ds \\ & \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_{n+1}}^{d_{n+2}} s^{N-2/(q-1)} e^{-s^2/4t} s\,ds. \end{aligned}$$

Using the fact that $N - 2/(q-1) \geq 0$, we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-2/(q-1)} e^{-s^2/4t} s\,ds \geq e^{-(n+2)/4} d_{n+1}^{N-2/(q-1)+1}(d_{n+2} - d_{n+1}) \tag{2.67}$$

$$\geq \frac{t}{4e^2} d_{n+1}^{N-2/(q-1)} e^{-n/4}. \tag{2.68}$$

Thus

$$\overline{u}_F(x,t) \leq \frac{C}{t^{1+N/2}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{1}{s}F \cap B_1\right) s\,ds, \tag{2.69}$$

which ends the proof. □

## 3 Estimate from below

If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)$, we denote $u_\mu = u_{\mu,0}$, that is the solution of

$$\left\{ \begin{aligned} \partial_t u_\mu - \Delta u_\mu + u_\mu^q = 0 \qquad & \text{in } Q_T, \\ u_\mu(.,0) = \mu \qquad & \text{in } \mathbb{R}^N. \end{aligned} \right. \tag{3.1}$$

The maximal $\sigma$-moderate solution of (1.1 ) which has an initial trace vanishing outside a closed set $F$ is defined by

$$\underline{u}_F = \sup \left\{ u_\mu : \mu \in \mathfrak{M}^q_+(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)\,,\ \mu(F^c) = 0 \right\}. \tag{3.2}$$

The main result of this section is the next one

**Theorem 3.1** *Assume $q \geq q_c$. There exists a constant $C_2 = C_2(N,q,T) > 0$ such that, for any closed subset $F \subset \mathbb{R}^N$, there holds*

$$\underline{u}_F(x,t) \geq C_2 W_F(x,t) \quad \forall (x,t) \in Q_T. \tag{3.3}$$

We first assume that $F$ is compact, and we shall denote it by $K$. The first observation is that if $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$, $u_\mu \in L^q(Q_T)$ (see lemma below) and $0 \leq u_\mu \leq \mathbb{H}[\mu] := \mathbb{H}_\mu$. Therefore

$$u_\mu \geq \mathbb{H}_\mu - \mathbb{G}\left[\mathbb{H}_\mu^q\right], \tag{3.4}$$

where $\mathbb{G}$ is the Green heat potential in $Q_T$ defined by

$$\mathbb{G}[f](t) = \int_0^t \mathbb{H}[f(s)](t-s)ds = \int_0^t \int_{\mathbb{R}^N} H(.,y,t-s) f(y,s) dy ds.$$

Since the details of the proof are very technical, we shall present its main line. The key idea is to construct, for any $(x,t) \in Q_T$, a measure $\mu = \mu(x,t) \in \mathfrak{M}^q_+(\mathbb{R}^N)$ such that there holds

$$\mathbb{H}_\mu(x,t) \geq C W_K(x,t) \quad \forall (x,t) \in Q_T, \tag{3.5}$$

and

$$\mathbb{G}\left(\mathbb{H}_\mu\right)^q \leq C\, \mathbb{H}_\mu \quad \text{in } Q_T, \tag{3.6}$$

with constants $C$ depends only on $N$, $q$, and $T$, then to replace $\mu$ by $\mu_\epsilon = \epsilon\mu$ with $\epsilon = (2C)^{-1/(q-1)}$ in order to derive

$$u_{\mu_\epsilon} \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \tag{3.7}$$

From this follows

$$\underline{u}_K \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \tag{3.8}$$

and the proof of Theorem 3.1 with $C_2 = 2^{-1}C$.

We recall the following regularity result which actually can be used for defining the norm in negative Besov spaces [30]

**Lemma 3.2** *There exists a constant $c > 0$ such that*

$$c^{-1}\|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}_\mu\|_{L^q(Q_T)} \leq c\|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \tag{3.9}$$

*for any $\mu \in W^{-2/q,q}(\mathbb{R}^N)$.*

## 3.1 Estimate from below for the heat equation

### 3.1.1 The extended slicing

If $K$ is a compact subset of $\mathbb{R}^N$, $m=(x,t)$, we define $d_K$, $\lambda$, $d_n$ and $a_t$ as in Section 2.3. Let $\alpha\in(0,1)$ to be fixed later on, we define $\mathcal{T}_n$ for $n\in\mathbb{Z}$ by

$$\mathcal{T}_n=\begin{cases}\mathcal{B}^2_{\sqrt{t(n+1)}}(m)\setminus\mathcal{B}^2_{\sqrt{tn}}(m) & \text{if } n\geq 1,\\ \mathcal{B}^2_{\alpha^{-n}\sqrt{t}}(m)\setminus\mathcal{B}^2_{\alpha^{1-n}\sqrt{t}}(m) & \text{if } n\leq 0,\end{cases}$$

and put

$$\mathcal{T}^*_n=\mathcal{T}_n\cap\{s:0\leq s\leq t\},\quad \text{for } n\in\mathbb{Z}.$$

We recall that for $n\in\mathbb{N}_*$,

$$\mathcal{Q}_n=K\cap\mathcal{B}^2_{\sqrt{t(n+1)}}(m)=K\cap B_{d_n}(x)$$

and

$$K_n=K\cap\mathcal{T}_{n+1}=K\cap\left(B_{d_{n+1}}(x)\setminus B_{d_n}(x)\right).$$

Let $\nu_n\in\mathfrak{M}_+(\mathbb{R}^N)\cap W^{-2/q,q}(\mathbb{R}^N)$ be the $q$-capacitary measure of the set $K_n/d_{n+1}$ (see [1, Sec. 2.2]). Such a measure has support in $K_n/d_{n+1}$ and

$$\nu_n(K_n/d_{n+1})=C_{2/q,q'}(K_n/d_{n+1}) \text{ and } \|\nu_n\|_{W^{-2/q,q'}(\mathbb{R}^N)}=\left(C_{2/q,q'}(K_n/d_{n+1})\right)^{1/q}. \tag{3.10}$$

We define $\mu_n$ as follows

$$\mu_n(A)=d_{n+1}^{N-2/(q-1)}\nu_n(A/d_{n+1})\quad \forall A\subset K_n,\ A \text{ Borel }, \tag{3.11}$$

and set

$$\mu_{t,K}=\sum_{n=0}^{a_t}\mu_n,$$

and

$$\mathbb{H}_{\mu_{t,K}}=\sum_{n=0}^{a_t}\mathbb{H}_{\mu_n} \tag{3.12}$$

**Proposition 3.3** *Let $q\geq q_c$, then there holds*

$$\mathbb{H}_{\mu_{t,K}}(x,t)\geq\frac{1}{(4\pi t)^{N/2}}\sum_{n=0}^{a_t}e^{-(n+1)/4}d_{n+1}^{N-2/(q-1)}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{3.13}$$

*in $\mathbb{R}^N\times(0,T)$.*

*Proof.* Since

$$\mathbb{H}_{\mu_n}(x,t)=\frac{1}{(4\pi t)^{N/2}}\int_{K_n}e^{-|x-y|^2/4t}d\mu_n, \tag{3.14}$$

and

$$y\in K_n\Longrightarrow|x-y|\leq d_{n+1},$$

(3.13 ) follows because of (3.11 ) and (3.12 ). ◻

### 3.2 Estimate from above for the nonlinear term

We write (3.4 ) under the form

$$
\begin{aligned}
u_\mu(x,t) &\geq \sum_{n\in\mathbb{Z}} \mathbb{H}_{\mu_n}(x,t) - \int_0^t \int_{\mathbb{R}^N} H(x,y,t-s) \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\
&= I_1 - I_2.
\end{aligned}
\tag{3.15}
$$

since $\mu_n = 0$ if $n \notin A_K = \mathbb{N} \cap [1, a_t]$, and

$$
\begin{aligned}
I_2 &\leq \frac{1}{(4\pi)^{N/2}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\
&\leq \frac{1}{(4\pi)^{N/2}} (J_\ell + J'_\ell),
\end{aligned}
\tag{3.16}
$$

for some $\ell \in \mathbb{N}^*$ to be fixed later on, where

$$
J_\ell = \sum_{p\in\mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,
$$

and

$$
J'_\ell = \sum_{p\in\mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[\sum_{n\geq p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds.
$$

The next estimate will be used several times in the sequel.

**Lemma 3.4** *Let $0 < a < b$ and $t > 0$, then,*

$$
\max\left\{\sigma^{-N/2} e^{-\rho^2/4\sigma} : 0 \leq \sigma \leq t,\ at \leq \rho^2 + \sigma \leq bt\right\} = e^{1/4} \begin{cases} t^{-N/2} e^{-a/4} & \text{if } \dfrac{a}{2N} > 1, \\[2ex] \left(\dfrac{2N}{at}\right)^{N/2} e^{-N/2} & \text{if } \dfrac{a}{2N} \leq 1. \end{cases}
$$

*Proof.* Set

$$
\mathcal{J}(\rho,\sigma) = \sigma^{-N/2} e^{-\rho^2/4\sigma}
$$

and

$$
\mathcal{K}_{a,b,t} = \left\{(\rho,\sigma) \in [0,\infty) \times (0,t] :\ at \leq \rho^2 + \sigma \leq bt\right\}.
$$

We first notice that, for fixed $\sigma$, the maximum of $\mathcal{J}(.,\sigma)$ is achieved for $\rho$ minimal. If $\sigma \in [at, bt]$ the minimal value of $\rho$ is 0, while if $\sigma \in (0, at)$, the minimum of $\rho$ is $\sqrt{at-s}$.

- Assume first $a \geq 1$, then $\mathcal{J}(\sqrt{at-\sigma},\sigma) = e^{1/4}\sigma^{-N/4} e^{-at/4\sigma}$, thus, if $1 \leq a/2N$ the minimal value of $\mathcal{J}(\sqrt{at-\sigma},\sigma)$ is $e^{(1-2N)/4}(2N/at)^{N/2}$, while, if $a/2N < 1 \leq a$, the minimum is $e^{1/4}t^{-N/2}e^{-a/4}$.

- Assume now $a \leq 1$. Then

$$\begin{aligned}\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}_{a,b,t}\} &= \max\left\{\max_{\sigma\in(at,t]}\mathcal{J}(0,\sigma),\ \max_{\sigma\in(0,at]}\mathcal{J}(\sqrt{at-\sigma},\sigma)\right\}\\ &= \max\left\{(at)^{-N/2},\ e^{(1-2N)/4}(2N/at)^{N/2}\right\}\\ &= e^{(1-2N)/4}(2N/at)^{N/2}.\end{aligned}$$

Combining these two estimates, we derive the result. □

*Remark.* The following variant of Lemma 3.4 will be useful in the sequel: *For any $\theta \geq 1/2N$ there holds*

$$\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}(a,b,t)\} \leq e^{1/4}\left(\frac{2N\theta}{t}\right)^{N/2} e^{-a/4} \quad \text{if } \theta a \geq 1. \tag{3.17}$$

**Lemma 3.5** *There exists a positive constant $C = C(N,\ell,q)$ such that*

$$J_\ell \leq Ct^{-N/2}\sum_{n=1}^{a_t} d_{n+1}^{N-2/(q-1)} e^{-(1+(n-\ell)_+)/4}\, C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.18}$$

*Proof.* The set of $p$ for the summation in $J_\ell$ is reduced to $\mathbb{Z}\cap[-\ell+2,\infty)$ and we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^{0}\iint_{\mathcal{T}_p^*}(t-s)^{-N/2}e^{-|x-y|^2/4(t-s)}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty}\iint_{\mathcal{T}_p^*}(t-s)^{-N/2}e^{-|x-y|^2/4(t-s)}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q.$$

If $p = 2-\ell,\ldots,0$,

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow t\alpha^{2-2p} \leq |x-y|^2 + t - s \leq t\alpha^{-2p},$$

and, if $p \geq 1$

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow pt \leq |x-y|^2 + t - s \leq (p+1)t.$$

By Lemma 3.4 and (3.17 ), there exists $C = C(N,\ell,\alpha) > 0$ such that

$$\max\left\{(t-s)^{-N/2}e^{-|x-y|^2/4(t-s)} : (y,s)\in\mathcal{T}_p^*\right\} \leq Ct^{-N/2}e^{-\alpha^{2-2p}/4}, \tag{3.19}$$

if $p = 2-\ell,\ldots,0$, and

$$\max\left\{(t-s)^{-N/2}e^{-|x-y|^2/4(t-s)} : (y,s)\in\mathcal{T}_p^*\right\} \leq Ct^{-N/2}e^{-p/4}, \tag{3.20}$$

if $p \geq 1$. When $p = 2 - \ell, \dots, 0$

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \leq C \sum_{1}^{p+\ell-1} \mathbb{H}^q_{\mu_n}(y,s). \tag{3.21}$$

for some $C = C(\ell, q) > 0$, thus

$$\begin{aligned} J_{1,\ell} &\leq Ct^{-N/2} \sum_{p=2-\ell}^{0} e^{-\alpha^{2-2p}/4} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\leq Ct^{-N/2} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \sum_{p=n-\ell+1}^{0} e^{-\alpha^{2-2p}/4} \\ &\leq Ct^{-N/2} e^{-\alpha^{2\ell-2}/4} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.22}$$

If the set of $p$'s is not upper bounded, we introduce $\delta > 0$ to be made precise later on. Then

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \leq \left[\sum_{1}^{p+\ell-1} e^{\delta q' n/4}\right]^{q/q'} \sum_{1}^{p+\ell-1} e^{-\delta q n/4} \mathbb{H}^q_{\mu_n}(y,s), \tag{3.23}$$

with $q' = q/(q-1)$. If, by convention $\mu_n = 0$ whenever $n > a_t$, we obtain, for some $C > 0$ which depends also on $\delta$,

$$\begin{aligned} J_{2,\ell} &\leq Ct^{-N/2} \sum_{p=1}^{\infty} e^{(\delta(p+\ell-1)q-p)/4} \sum_{n=1}^{p+\ell-1} e^{-\delta q n/4} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\leq Ct^{-N/2} \sum_{n=1}^{\infty} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} e^{-\delta q n/4} \sum_{p=(n-\ell+1)\vee 1}^{\infty} e^{(\delta(p+\ell-1)q-p)/4} \\ &\leq Ct^{-N/2} \sum_{n=1}^{\infty} e^{-(1+(n-\ell)_+)/4} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.24}$$

Notice that we choose $\delta$ such that $\delta \ell q < 1$. Combining (3.22 ) and (3.24 ), we derive (3.18 ) from Lemma 3.2, (3.10 ) and (3.11 ). $\square$

The set of indices $p$ for which the $\mu_n$ terms are not zero in $J'_\ell$ is $\mathbb{Z} \cap (-\infty, a_t - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^{0} \iint_{\mathcal{T}^*_p} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[ \sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds.$$

**Lemma 3.6** *There exists a constant $C = C(N,q,\ell) > 0$ such that*

$$J'_{1,\ell} \leq Ct^{1-Nq/2} \sum_{n=0}^{a_t} e^{-(1+\beta_0)(n-h)_+/4} d_{n+1}^{Nq-2q'} C_{2/q,q'}^q \left( \frac{K_n}{d_{n+1}} \right), \tag{3.25}$$

*where $\beta_0 = (q-1)/4$ and $h = 2q(q+1)/(q-1)^2$.*

*Proof.* Since

$$(y,s) \in \mathcal{T}_p^*, \text{ and } (z,0) \in K_n \Longrightarrow |y-z| \geq (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{3.26}$$

there holds

$$\mathbb{H}_{\mu_n}(y,s) \leq (4\pi s)^{-N/2} e^{-(\sqrt{n}-\alpha^{-p})^2 t/4s} \mu_n(K_n) \leq Ct^{-N/2} e^{-(\sqrt{n}-\alpha^{-p})^2/4} \mu_n(K_n),$$

by Lemma 3.4. Let $\epsilon_n > 0$ such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$\begin{aligned}
J'_{1,\ell} &\leq CA_\epsilon^{q/q'} t^{-Nq/2} \sum_{p=-\infty}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \sum_{n=1\vee(p+\ell)}^{\infty} \epsilon_n^{-q} e^{-q(\sqrt{n}-\alpha^{-p})^2/4} \mu_n^q(K_n) ds\, dy \\
&\leq CA_\epsilon^{q/q'} t^{-Nq/2} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) \sum_{-\infty}^{p=0\wedge(n-\ell)} e^{-q(\sqrt{n}-\alpha^{-p})^2/4} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} ds\, dy \\
&\leq CA_\epsilon^{q/q'} t^{-Nq/2} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-q(\sqrt{n}-1)^2/4} \iint_{\cup_{p\leq 0}\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} ds\, dy \\
&\leq CA_\epsilon^{q/q'} t^{1-Nq/2} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-q(\sqrt{n}-1)^2/4}.
\end{aligned} \tag{3.27}$$

Set $h = 2q(q+1)/(q-1)^2$ and $Q = (1+q)/2$, then $q(\sqrt{n}-1)^2 \geq Q(n-h)_+$ for any $n \geq 1$. If we choose $\epsilon_n = e^{-(q-1)(n-h)_+/16q}$, there holds $\epsilon_n^{-q} e^{-q(\sqrt{n}-1)^2/4} \leq e^{(q+3)(n-h)_+/16}$. Finally

$$J'_{1,\ell} \leq Ct^{1-Nq/2} \sum_{n=1}^{\infty} e^{(1+\epsilon_0)(n-h)_+/4} \mu_n^q(K_n),$$

with $\beta_0 = (q-1)/4$, which yields to (3.25 ) by the choice of the $\mu_n$. $\square$

In order to make easier the obtention of the estimate of the term $J'_{2,\ell}$, we first give the proof in dimension 1.

**Lemma 3.7** *Assume $N = 1$ and $\ell$ is an integer larger than 1. There exists a positive constant $C = C(q,\ell) > 0$ such that*

$$J'_{2,\ell} \leq Ct^{-1/2}\sum_{n=\ell}^{a_t} e^{-n/4} d_{n+1}^{(q-3)/(q-1)} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.28}$$

*Proof.* If $(y,s) \in \mathcal{T}_p^*$ and $z \in K_n$ ($p \geq 1$, $n \geq p = \ell$) , there holds $|x-y| \geq \sqrt{t}\sqrt{p}$ and $|y-z| \geq \sqrt{t}(\sqrt{n}-\sqrt{p+1})$. Therefore

$$J'_{2,\ell} \leq C\sqrt{t}\sum_{p=1}^{a_t-\ell}\frac{1}{\sqrt{p}}\int_0^t e^{-pt/4(t-s)}\left(\sum_{n=p+\ell}^{a_t} s^{-1/2}e^{-(\sqrt{n}-\sqrt{p+1})^2t/4s}\mu_n(K_n)\right)^q.$$

If $\epsilon \in (0,q)$ is some positive parameter which will be made more precise later on, there holds

$$\begin{aligned}&\left(\sum_{n=p+\ell}^{a_t} s^{-1/2}e^{-(\sqrt{n}-\sqrt{p+1})^2t/4s}\mu_n(K_n)\right)^q \\ &\qquad\leq \left(\sum_{n=p+\ell}^{a_t} e^{-\epsilon q'(\sqrt{n}-\sqrt{p+1})^2t/4s}\right)^{q/q'}\sum_{n=p+\ell}^{a_t} s^{-q/2}e^{-(q-\epsilon)(\sqrt{n}-\sqrt{p+1})^2)t/4s}\mu_n^q(K_n),\end{aligned}$$

by Hölder's inequality. By comparison between series and integrals and using Gauss' integral

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} e^{-\epsilon q'(\sqrt{n}-\sqrt{p+1})^2t/4s} &\leq \int_{p+\ell}^\infty e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2t/4s}dx \\ &= 2\int_{\sqrt{p+\ell}-\sqrt{p+1}}^\infty e^{-\epsilon q'x^2t/4s}(x+\sqrt{p+1})dx \\ &\leq \frac{4s}{\epsilon q't}e^{-\epsilon q'(\sqrt{p+\ell}-\sqrt{p+1})^2t/4s} + 2\sqrt{p+1}\int_{\sqrt{p+\ell}-\sqrt{p+1}}^\infty e^{-\epsilon q'x^2t/4s}dx \\ &\leq C\sqrt{\frac{(p+1)s}{t}}e^{-\epsilon q'(\sqrt{p+\ell}-\sqrt{p+1})^2t/2s} \\ &\leq C\sqrt{\frac{(p+1)s}{t}}.\end{aligned}$$

If we set $q_\epsilon = q-\epsilon$, then

$$J'_{2,\ell} \leq C\epsilon^{-q'/q}t^{1-q/2}\sum_{n=\ell+1}^{\infty}\mu_n^q(K_n)\sum_{p=1}^{n-\ell}p^{(q-2)/2}\int_0^t(t-s)^{-1/2}s^{-1/2}e^{-pt/4(t-s)}e^{-q_\epsilon(\sqrt{n}-\sqrt{p+1})^2)t/4s}ds.$$

where $C = C(\epsilon,q) > 0$. Since

$$\begin{aligned}&\int_0^t(t-s)^{-1/2}s^{-1/2}e^{-pt/4(t-s)}e^{-q_\epsilon(\sqrt{n}-\sqrt{p+1})^2)t/4s}ds \\ &\qquad\qquad = \int_0^1(1-s)^{-1/2}s^{-1/2}e^{-p/4(1-s)}e^{-q_\epsilon(\sqrt{n}-\sqrt{p+1})^2/4s}ds,\end{aligned}$$

we can apply Lemma A.1 with $a = 1/2$, $b = 1/2$, $A = \sqrt{p}$ and $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$. In this range of indices $B \geq \sqrt{q_\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \sqrt{q_\epsilon}(\ell-1)\sqrt{p}$, thus $\kappa = \sqrt{q_\epsilon}(\ell-1)$ and

$$\sqrt{\frac{A}{A+B}}\sqrt{\frac{B}{A+B}} \leq p^{1/4}n^{-1/2}(\sqrt{n}-\sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2}s^{-q/2}e^{-pt/4(t-s)}e^{-q(\sqrt{n}-\sqrt{p+1})^2t/4s}ds \leq \frac{Cp^{1/4}(\sqrt{n}-\sqrt{p})^{1/2}}{\sqrt{n}}e^{-(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2/4}, \tag{3.29}$$

which implies

$$J'_{2,\ell} \leq Ct^{1-q/2}\sum_{n=\ell+1}^{a_t}\frac{\mu_n^q(K_n)}{\sqrt{n}}\sum_{p=1}^{n-\ell}p^{(2q-3)/4}(\sqrt{n}-\sqrt{p})^{1/2}e^{-(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2/4}, \tag{3.30}$$

where $C$ depends of $\epsilon$, $q$ and $\ell$. By Lemma A.2

$$J'_{2,\ell} \leq Ct^{1-q/2}\sum_{n=\ell+1}^{a_t}n^{(q-3)/2}e^{-n/4}\mu_n^q(K_n) \tag{3.31}$$

Because $\mu_n(K_n) = d_{n+1}^{(q-3)/(q-1)}C_{2/q,q'}(K_n/d_{n+1})$ (remember $N=1$) and $\operatorname{diam} K_n/d_{n+1} \leq 1/n$, there holds

$$\mu_n^q(K_n) \leq C(\sqrt{t}/\sqrt{n})^{q-3}\mu_n(K_n) = C(\sqrt{t}/\sqrt{n})^{q-3}d_{n+1}^{(q-3)/(q-1)}C_{2/q,q'}(K_n/d_{n+1}) \tag{3.32}$$

and inequality (3.28 ) follows. □

Next we give the general proof. For this task we shall use again the quasi-additivity with separated partitions.

**Lemma 3.8** *Assume $N \geq 2$ and $\ell$ is an integer larger than 1. There exist a positive constant $C_1 = C_1(q,N,\ell) > 0$ such that f*

$$J'_{2,\ell} \leq C_1t^{-N/2}\sum_{n=\ell}^{a_t}e^{-n/4}d_{n+1}^{N-2/(q-1)}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.33}$$

*Proof.* As in the proof of Theorem 2.14, we know that there exists a finite number $J$, depending only on the dimension $N$, of separated sub-partitions $\{\#\Theta_{t,n}^h\}_{h=1}^J$ of the sets $T_n$ by the $N$-dim balls $B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})$ where $|a_{n,j}| = (d_{n+1}+d_n)/2$ and $|a_{n,j}-a_{n,k}| \geq \sqrt{t}/2\sqrt{n+1}$. Furthermore $\#\Theta_{t,n}^h \leq Cn^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})$. We write $\mu_n = \sum_{h=1}^J \mu_n^h$, and accordingly $J'_{2,\ell} = \sum_{h=1}^J {J'}_{2,\ell}^h$, where $\mu_n^h = \sum_{j\in\Theta_{t,n}^h}\mu_{n,j}$, and $\mu_{n,j}$ are the capacitary measures of $K_{n,j}$ relative to $B_{n,j} = B_{6t/5\sqrt{n}}(a_n,j)$, which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j})\right)^{1/q}. \tag{3.34}$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[ \sum_{n=p+\ell}^{\infty} \sum_{h=1}^{J} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s) \right]^q dyds.$$

We denote

$$J'^{\,h}_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-N/2} e^{-|x-y|^2/4(t-s)} \left[ \sum_{n=p+\ell}^{\infty} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s) \right]^q dyds,$$

and clearly

$$J'_{2,\ell} \le C \sum_{h=1}^{J} J'^{\,h}_{2,\ell}, \tag{3.35}$$

where $C$ depends only on $N$ and $q$. For integers $n$ and $p$ such that $n \ge \ell+1$, we set

$$\lambda_{n,j,y} = \inf\{|y-z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y-a_{n,j}| - \sqrt{t}/\sqrt{n+1}.$$

Therefore

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \int_{K_n} e^{-|y-z|^2/4s} d\mu_n^h(z) &= \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} \int_{K_{n,j}} e^{-|y-z|^2/4s} d\mu_{n,j}(z) \\ &\le \left( \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-\epsilon q' \lambda_{n,j,y}^2/4s} \right)^{1/q'} \left( \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-q\lambda_{n,j,y}^2(1-\epsilon)/4s} \mu_{n,j}^q(K_{n,j}) \right)^{1/q}\end{aligned}$$

where $\epsilon > 0$ will be made precise later on.

*Step 1* We claim that

$$\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\epsilon q' \lambda_{n,j,y}^2/4s} \le C\sqrt{\frac{ps}{t}} \tag{3.36}$$

where $C$ depends on $\epsilon$, $q$ and $N$. If $y$ is fixed in $T_p$, we denote by $z_y$ the point of $T_n$ which solves $|y-z_y| = \operatorname{dist}(y, T_n)$. Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \le |y - z_y| \le t(\sqrt{n} - \sqrt{p}).$$

Let $Y = y\sqrt{t(p+1)}/|y|$. On the axis $\overrightarrow{0Y}$ we set $\mathbf{e} = Y/|Y|$, consider the points $b_k = (k\sqrt{t}/\sqrt{n})\mathbf{e}$ where $-n \le k \le n$ and denote by $G_{n,k}$ the spherical shell obtain by intersecting the spherical shell $T_n$ with the domain $H_{n,k}$ which is the set of points in $\mathbb{R}^N$ limited by the hyperplanes orthogonal to $\overrightarrow{0Y}$ going through $((k+1)\sqrt{t}/\sqrt{n})\mathbf{e}$ and $((k-1)\sqrt{t}/\sqrt{n})\mathbf{e}$. The number of points $a_{n,j} \in G_{n,k}$ is smaller than $C(n+1-|k|)^{N-2}$, where $C$ depends only on $N$, and we denote by $\Lambda_{n,k}$ the set of $j \in \Theta_{t,n}$ such that $a_{n,j} \in G_{n,k}$. Furthermore, if $a_{n,j} \in G_{n,k}$ elementary geometric

considerations (Pythagore's theorem) imply that $\lambda^2_{n,j,y}$ is greater than $t(n+p+1-2k\sqrt{p+1}/\sqrt{n})$. Therefore

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\lambda^2_{n,j,y}/4s} \le C\sum_{n=p+\ell}^{a_t}\sum_{k=-n}^{n}(n+1-|k|)^{N-2}e^{-\epsilon q'\left(n+p+1-2k\sqrt{p+1}/\right)t/4s\sqrt{n}} \tag{3.37}$$

*Case* $N=2$. By summing a geometric series and using the inequality $e^u/(e^u-1)\le 1+1/u$ for $u>0$, we obtain

$$\begin{aligned}\sum_{k=-n}^{n} e^{\epsilon q'\left(k\sqrt{p+1}/\right)t/2s\sqrt{n}} &\le e^{\epsilon q' t\sqrt{n(p+1)}/2s}\frac{e^{\epsilon q' t\sqrt{p+1}/2s\sqrt{n}}}{e^{\epsilon q' t\sqrt{p+1}/2s\sqrt{n}}-1}\\ &\le e^{\epsilon q' t\sqrt{n(p+1)}/2s}\left(1+\frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right).\end{aligned} \tag{3.38}$$

Thus, by comparison between series and integrals,

$$\begin{aligned}\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\lambda^2_{n,j,y}/4s} &\le C\sum_{n=p+\ell}^{a_t}\left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right)e^{-\epsilon q'(\sqrt{n}-\sqrt{p+1})^2 t/4s}\\ &\le C\int_{p+1}^{\infty} e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t/4s}dx\\ &\qquad + \frac{Cs}{t\sqrt{p}}\int_{p+1}^{\infty}\sqrt{x}e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t/4s}dx.\end{aligned} \tag{3.39}$$

Next

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t/4s}dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\epsilon q'(y-\sqrt{p+1})^2 t/4s}y\,dy\\ &= 2\int_0^{\infty} e^{-\epsilon q' y^2 t/4s}y\,dy + 2\sqrt{p+1}\int_0^{\infty} e^{-\epsilon q' y^2 t/4s}dy\\ &= \frac{2s}{t}\int_0^{\infty} e^{-\epsilon q' z^2/4}z\,dz + 2\sqrt{\frac{(p+1)s}{t}}\int_0^{\infty} e^{-\epsilon q' z^2/4}dz,\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}\int_{p+1}^{\infty}\sqrt{x}e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t/4s}dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\epsilon q'(y-\sqrt{p+1})^2 t/4s}y^2dy\\ &= 2\int_0^{\infty} e^{-\epsilon q' y^2 t/4s}(y+\sqrt{p+1})^2dy\\ &\le 4\int_0^{\infty} e^{-\epsilon q' y^2 t/4s}y^2dy + 4(p+1)\int_0^{\infty} e^{-\epsilon q' y^2 t/4s}dy\\ &\le 4\left(\frac{s}{t}\right)^{3/2}\int_0^{\infty} e^{-\epsilon q' z^2/4}z^2dz + 4(p+1)\sqrt{\frac{s}{t}}\int_0^{\infty} e^{-\epsilon q' z^2/4}dz\end{aligned} \tag{3.41}$$

Jointly with (3.39 ), these inequalities imply

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\lambda^2_{n,j,y}/4s} \le C\sqrt{\frac{ps}{t}} \tag{3.42}$$

*Case* $N>2$ Because the value of the right-hand side of (3.37 ) is an increasing value of $N$, it is sufficient to prove (3.36 ) when $N$ is even, say $(N-2)/2 = d \in \mathbb{N}_*$. There holds

$$\sum_{k=-n}^{n}(n+1-|k|)^d e^{\epsilon q'\left(k\sqrt{p+1}/\right)t/2s\sqrt{n}} \le 2\sum_{k=0}^{n}(n+1-k)^d e^{\epsilon q'\left(k\sqrt{p+1}/\right)t/2s\sqrt{n}} \tag{3.43}$$

We set

$$\alpha = \epsilon q'\left(\sqrt{p+1}/\right)t/2s\sqrt{n} \quad \text{ and } \quad I_d = \sum_{k=0}^{n}(n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha} = \frac{e^{(k+1)\alpha}-e^{k\alpha}}{e^{\alpha}-1}$$

we use Abel's transform to obtain

$$\begin{aligned} I_d &= \frac{1}{e^{\alpha}-1}\left(e^{(n+1)\alpha}-(n+1)^d+\sum_{k=1}^{n}\left((n+2-k)^d-(n+1-k)^d\right)e^{k\alpha}\right) \\ &\le \frac{1}{e^{\alpha}-1}\left((1-d)e^{(n+1)\alpha}-(n+1)^d+de^{\alpha}\sum_{k=1}^{n}\left((n+1-k)^{d-1}\right)e^{k\alpha}\right). \end{aligned}$$

Therefore the following induction holds

$$I_d \le \frac{de^{\alpha}}{e^{\alpha}-1}I_{d-1}. \tag{3.44}$$

In (3.38 ), we have already used the fact that

$$\frac{de^{\alpha}}{e^{\alpha}-1} \le C\left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right),$$

and

$$I_d \le C\left(1+\left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right)I_0.$$

Thus (3.39 ) is replaced by

$$\begin{aligned} \sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\lambda^2_{n,j,y}/4s} &\le C\sum_{n=p+\ell}^{a_t}\left(1+\left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right)e^{-\epsilon q'(\sqrt{n}-\sqrt{p+1})^2t/4s} \\ &\le C\int_{p+1}^{\infty}e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2t/4s}dx \\ &\qquad + \left(\frac{Cs}{t\sqrt{p}}\right)^{d+1}\int_{p+1}^{\infty}x^{(d+1)/2}e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2t/4s}dx. \end{aligned} \tag{3.45}$$

The first integral on the right-hand side has already been estimated in (3.40 ), for the second integral, there holds

$$\begin{aligned}\int_{p+1}^{\infty} x^{(d+1)/2} e^{-\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t/4s} dx &= \int_0^{\infty} (y+\sqrt{p+1}\,)^{d+2} e^{-\epsilon q' y^2 t/4s} dx \\ &\leq C\int_0^{\infty} y^{d+2} e^{-\epsilon q' y^2 t/4s} dy + Cp^{(d+2)/2}\int_0^{\infty} e^{-\epsilon q' y^2 t/4s} dy \\ &\leq C\left(\frac{s}{t}\right)^{2+d/2}\int_0^{\infty} z^{(d+1)/2} e^{-\epsilon q' z^2/4} dz \\ &\qquad + C\left(\frac{s}{t}\right)^{3/2} p^{(d+2)/2}\int_0^{\infty} e^{-\epsilon q' z^2/4} dz.\end{aligned} \tag{3.46}$$

Combining (3.40 ), (3.45 )) and (3.46 ), we derive (3.36 ).

*Step 2* Since $\mathcal{T}_p^* \subset \Gamma_p \times [0,t]$ where $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$, $(y,s) \in \mathcal{T}_p^*$ implies that $|x-y|^2 \geq (p-1)t$, thus $J'^{\,h}_{2,\ell}$ satisfies

$$\begin{aligned}J'^{\,h}_{2,\ell} &\leq Ct^{(1-q)/2}\sum_{p=1}^{\infty} p^{(q-1)/2}\int_0^t\int_{\Gamma_p}(t-s)^{-N/2}s^{-(q(N-1)+1)/2}e^{-|x-y|^2/4(t-s)} \\ &\qquad\qquad \times \sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta^h_{t,n}} e^{-q\lambda^2_{n,j,y}(1-\epsilon)/4s}\mu^q_{n,j}(K_{n,j})dsdy \\ &\leq Ct^{(1-q)/2}\sum_{n=\ell+1}^{a_t}\sum_{j\in\Theta^h_{t,n}}\mu^q_{n,j}(K_{n,j}) \\ &\quad \times \sum_{p=1}^{n-\ell} p^{(q-1)/2}\int_0^t\int_{\Gamma_p}(t-s)^{-N/2}s^{-(q(N-1)+1)/2}e^{-|x-y|^2/4(t-s)}e^{-q\lambda^2_{n,j,y}(1-\epsilon)/4s}dsdy\end{aligned} \tag{3.47}$$

and the constant $C$ depends on $N, q$ and $\epsilon$. Next we set $q_\epsilon = (1-\epsilon)q$. Writting

$$|y-a_{n,j}|^2 = |x-y|^2 + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle \geq pt + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle,$$

we get

$$\int_{\Gamma_p} e^{-q_\epsilon|y-a_{n,j}|^2/4s}dy = e^{-q_\epsilon|x-a_{n,j}|^2/4s}\int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-q_\epsilon r^2/4s}\int_{|x-y|=r} e^{2q_\epsilon\langle y-x, a_{n,j}-x\rangle/4s}dS_r(y)dr.$$

For estimating the value of the spherical integral, we can assume that $a_{n,j}-x = (0,\ldots,0,|a_{n,j}-x|)$, $y=(y_1,\ldots,y_N)$ and, using spherical coordinates with center at $x$, that the unit sphere has the representation $S^{N-1} = \{(\sin\phi.\sigma, \cos\phi) \in \mathbb{R}^{N-1}\times\mathbb{R} : \sigma \in S^{N-2},\ \phi\in[0,\pi]\}$. With this representation, $dS_r = r^{N-1}\sin^{N-2}\phi\, d\phi\, d\sigma$ and $\langle y-x, a_{n,j}-x\rangle = |a_{n,j}-x|\,|y-x|\cos\phi$. Therefore

$$\int_{|x-y|=r} e^{2q_\epsilon\langle y-x, a_{n,j}-x\rangle/4s}dS_r(y) = r^{N-1}\left|S^{N-2}\right|\int_0^{\pi} e^{2q_\epsilon|a_{n,j}-x|r\cos\phi/4s}\sin^{N-2}\phi\, d\phi.$$

By Lemma A.3

$$\begin{aligned}\int_{|x-y|=r} e^{2q_\epsilon\langle y-x,a_{n,j}-x\rangle/4s}dS_r(y) &\le C\frac{r^{N-1}e^{2q_\epsilon r|a_{n,j}-x|/4s}}{(1+r\,|a_{n,j}-x|\,/s)^{(N-1)/2}}\\ &\le Cs^{(N-1)/2}\left(\frac{r}{|a_{n,j}-x|}\right)^{(N-1)/2}e^{2q_\epsilon r|a_{n,j}-x|/4s}.\end{aligned}\tag{3.48}$$

Therefore

$$\int_{\Gamma_p} e^{-q_\epsilon|y-a_{n,j}|^2/4s}dy \le Ct^{(N+1)/4}p^{(N-3)/4}\frac{s^{(N-1)/2}e^{-q_\epsilon(|a_{n,j}-x|-\sqrt{t(p+1)})^2/4s}}{|a_{n,j}-x|^{(N-1)/2}},\tag{3.49}$$

and, since $|a_{n,j}-x|\ge\sqrt{tn}$,

$$\begin{aligned}&\int_0^t\int_{\Gamma_p}(t-s)^{-N/2}s^{-(q(N-1)+1)/2}e^{-|x-y|^2/4(t-s)}e^{-q_\epsilon\lambda^2_{n,j,y}/4s}dy\,ds\\ &\quad\le C\frac{\sqrt{t}p^{(N-3)/4}}{n^{(N-1)/4}}\int_0^t(t-s)^{-N/2}s^{-((q-1)(N-1)+1)/2}e^{-pt/4(t-s)}e^{-q_\epsilon(\sqrt{tn}-\sqrt{t(p+1)})^2/4s}ds\\ &\quad\le C\frac{t^{(1-q(N-1))/2}p^{(N-3)/4}}{n^{(N-1)/4}}\int_0^1(1-s)^{-N/2}s^{-((q-1)(N-1)+1)/2}e^{-p/4(1-s)}e^{-q_\epsilon(\sqrt{n}-\sqrt{p+1})^2/4s}.\end{aligned}\tag{3.50}$$

We apply Lemma A.1, with $A=\sqrt{p}$, $B=\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1})$, $b=((q-1)(N-1)+1)/2$, $a=N/2$ and $\kappa=\sqrt{q_\epsilon}(\ell-1)/8$ as in the case $N=1$, and noticing that, for these specific values,

$$\begin{aligned}A^{1-a}B^{1-b}(A+B)^{a+b-2} &= p^{(2-N)/4}(\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{(1-(q-1)(N-1)/2}\\ &\qquad\times(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{((q-1)(N-1)+N-3)/2}\\ &\le C\left(\frac{n}{p}\right)^{N/4-1/2}\left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{(1-(q-1)(N-1)/2},\end{aligned}$$

where $C$ depends on $N$, $q$ and $\kappa$. Therefore

$$\begin{aligned}&\int_0^t\int_{\Gamma_p}(t-s)^{-N/2}s^{-N/2}e^{-|x-y|^2/4(t-s)}e^{-q_\epsilon|y-z|^2/4s}dy\,ds\\ &\quad\le C\frac{t^{(1-q(N-1))/2}p^{(N-3)/4}}{n^{(N-1)/4}}\left(\frac{n}{p}\right)^{N/4-1/2}\left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{(1-(q-1)(N-1)/2}e^{-(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2/4}\\ &\quad\le Ct^{(1-q(N-1))/2}p^{-1/4}n^{((q-1)(N-1)-2)/4}(\sqrt{n}-\sqrt{p})^{(1-(q-1)(N-1)/2}e^{-(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2/4}.\end{aligned}\tag{3.51}$$

We derive from (3.47 ), (3.51 ),

$$\begin{aligned}&J'^{\,h}_{2,\ell}\le Ct^{1-Nq/2}\\ &\times\sum_{n=\ell+1}^{a_t}\sum_{j\in\Theta^h_{t,n}}n^{((q-1)(N-1)-2)/4}\mu^q_{n,j}(K_{n,j})\sum_{p=1}^{n-\ell}p^{(2q-3)/4}(\sqrt{n}-\sqrt{p})^{(1-(q-1)(N-1)/2}e^{-(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2/4}.\end{aligned}\tag{3.52}$$

By Lemma A.2 with $\alpha = (2q-3)/4$, $\beta = (1-(q-1)(N-1)/2$, $\delta = 1/4$ and $\gamma = q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell} p^{(2q-3)/4}(\sqrt{n}-\sqrt{p})^{(1-(q-1)(N-1)/2} e^{-(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2/4} \leq Cn^{(N(q-1)+q-3)/4} e^{-n/4}, \tag{3.53}$$

thus

$$J'^{\,h}_{2,\ell} \leq Ct^{1-Nq/2} \sum_{n=\ell+1}^{a_t} n^{N(q-1)/2-1} e^{-n/4} \sum_{j\in\Theta^h_{t,n}} \mu^q_{n,j}(K_{n,j}). \tag{3.54}$$

Because

$$\mu_{n,j}(K_{n,j}) = C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \approx \left(\frac{t}{n+1}\right)^{N/2-1/(q-1)} C_{2/q,q'}(\sqrt{n+1}K_{n,j}/\sqrt{t})$$

and diam $(\sqrt{n+1}K_{n,j}/\sqrt{t}) \leq 2$, there holds

$$\mu^q_{n,j}(K_{n,j}) \leq \left(\frac{t}{n}\right)^{N(q-1)/2-1} C^{B_{n,j}}_{2/q,q'}(K_{n,j}), \tag{3.55}$$

we obtain

$$\begin{aligned} J'^{\,h}_{2,\ell} &\leq Ct^{-N/2} \sum_{n=\ell+1}^{a_t} e^{-n/4} \sum_{j\in\Theta^h_{t,n}} C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \\ &\leq Ct^{-N/2} \sum_{n=\ell+1}^{a_t} e^{-n/4} \left(\frac{t}{n}\right)^{N/2-1/(q-1)} C_{2/q,q'}(\sqrt{n}K_n/\sqrt{t}). \end{aligned} \tag{3.56}$$

by using (2.52 ) in Lemma 2.15. Since $C_{2/q,q'}(\sqrt{n}K_n/\sqrt{t}) \leq (d_{n+1}\sqrt{n}/\sqrt{t})^{N-2/(q-1)} C_{2/q,q'}(K_n/d_{n+1})$, we finally derive

$$J'^{\,h}_{2,\ell} \leq Ct^{-N/2} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-2/(q-1)} e^{-n/4} \sum_{j\in\Theta^h_{t,n}} \mu^q_{n,j}(K_{n,j}). \tag{3.57}$$

Using again the quasi-additivity and the fact that $J'_{2,\ell} = \sum_{h=1}^{J} J'^{\,h}_{2,\ell}$, we deduce

$$J_{2,\ell} \leq C't^{-N/2} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-2/(q-1)} e^{-n/4} C_{2/q,q'}(K_n/d_{n+1}), \tag{3.58}$$

which implies (3.33 ). $\square$

The proof of Theorem 3.1 follows from the previous estimates on $J_1$ and $J_2$. Furthermore the following integral expression holds

**Theorem 3.9** *Assume $q \geq q_c$. Then there exists a positive constants $C_2^*$ , depending on $N$,$q$ and $T$, such that for any closed set $F$, there holds*

$$\underline{u}_F(x,t) \geq \frac{C_2^*}{t^{1+N/2}} \int_0^{\sqrt{ta_t}} e^{-s^2/4t} s^{N-2/(q-1)} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds, \tag{3.59}$$

*where $a_t$ is the smallest integer $j$ such that $F \subset B_{\sqrt{jt}}(x)$.*

*Proof.* We shall distinguish according $q = q_c$, or $q > q_c$, and for simplicity we shall denote $B_r = B_r(x)$ for the various values of $r$.

*Case 1: $q = q_c \Longleftrightarrow N - 2/(q-1) = 0$.* Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right) = C_{2/q,q'}\left(\frac{d_n}{d_{n+1}} \frac{F \cap B_{d_n}}{d_n}\right) \leq C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

thus

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

it follows

$$\begin{aligned}
\sum_{n=1}^{a_t} e^{-n/4} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) &\geq \sum_{n=1}^{a_t} e^{-n/4} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - \sum_{n=1}^{a_t} e^{-n/4} C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right) \\
&\geq \sum_{n=1}^{a_t} e^{-n/4} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-1/4} \sum_{n=0}^{a_t-1} e^{-n/4} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \\
&\geq (1 - e^{-1/4}) \sum_{n=1}^{a_t-1} e^{-n/4} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-1/4} C_{2/q,q'}\left(\frac{F}{\sqrt{t}} \cap B_1\right).
\end{aligned}$$

Since, by (2.66 ),

$$C_{2/q,q'}\left(\frac{F}{s'} \cap B_1\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \geq C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right),$$

for any $s' \in [d_{n+1}, d_{n+2}]$ and $s \in [d_n, d_{n+1}]$, there holds

$$\begin{aligned}
te^{-n/4} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) &\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_n}^{d_{n+1}} e^{-s^2/4t} s\, ds \\
&\geq \int_{d_n}^{d_{n+1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\, ds.
\end{aligned}$$

This implies

$$W_F(x,t) \geq (1 - e^{-1/4}) t^{-(1+N/2)} \int_0^{\sqrt{ta_t}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\, ds.$$

*Case 2:* $q > q_c \iff N - 2/(q-1) > 0$. In that case it is known [1] that

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \approx d_{n+1}^{2/(q-1)-N} C_{2/q,q'}(F_n)$$

thus

$$W_F(x,t) \approx t^{-1-N/2} \sum_{n=0}^{a_t} e^{-n/4} C_{2/q,q'}(F_n).$$

Since

$$C_{2/q,q'}(F_n) \geq C_{2/q,q'}\left(F \cap B_{d_{n+1}}\right) - C_{2/q,q'}\left(F \cap B_{d_n}\right),$$

and again

$$\begin{aligned} t^{-N/2} \sum_{n=0}^{a_t} e^{-n/4} C_{2/q,q'}(F_n) &\geq (1 - e^{-1/4}) t^{-N/2} \sum_{n=0}^{a_t - 1} e^{-n/4} C_{2/q,q'}\left(F \cap B_{d_{n+1}}\right) \\ &\geq (1 - e^{-1/4}) t^{-(1+N/2)} \int_0^{\sqrt{t a_t}} e^{-s^2/4t} C_{2/q,q'}(F \cap B_s)\, s\, ds. \end{aligned}$$

Because $C_{2/q,q'}(F \cap B_s) \approx s^{N-2/(q-1)} C_{2/q,q'}\left(s^{-1} F \cap B_1\right)$, (3.59 ) follows. $\square$

# 4 Applications

The first result of this section is the following

**Theorem 4.1** *Assume $N \geq 1$ and $q > 1$. Then $\overline{u}_K = \underline{u}_K$.*

*Proof.* If $1 < q < q_c$, the result is already proved in [21]. The proof in the super-critical case is an adaptation that we shall recall, for the sake of completeness. By Theorem 2.16 and Theorem 3.9 there exists a positive constant $C$, depending on $N$, $q$ and $T$ such that

$$\overline{u}_F(x,t) \leq \underline{u}_F(x,t) \quad \forall (x,t) \in Q_T.$$

By convexity $\tilde{u} = \underline{u}_F - \dfrac{1}{2C}(\overline{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than $\underline{u}_F$ if we assume that $\overline{u}_F \neq \underline{u}_F$. If we set $\theta := 1/2 + 1/(2C)$, then $u_\theta = \theta \overline{u}_F$ is a subsolution. Therefore there exists a solution $u_1$ of (1.1 ) in $Q_\infty$ such that $u_\theta \leq u_1 \leq \tilde{u} < \underline{u}_F$. If $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N)$ satisfies $\mu(F^c) = 0$, then $u_{\theta\mu}$ is the smallest solution of (1.1 ) which is above the subsolution $\theta u_\mu$. Thus $u_{\theta\mu} \leq u_1 < \underline{u}_F$ and finally $\underline{u}_F \leq u_1 < \underline{u}_F$, a contradiction. $\square$

If we combine Theorem 2.16 and Theorem 3.9 we derive the following integral approximation of the capacitary potential

**Proposition 4.2** *Assume $q \geq q_c$. Then there exist two positive constants $C_1^\dagger$, $C_2^\dagger$, depending only on $N$, $q$ and $T$ such that*

$$\begin{aligned} C_2^\dagger t^{-(1+N/2)} \int_0^{\sqrt{ta_t}} s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \leq W_F(x,t) \\ \leq C_1^\dagger t^{-(1+N/2)} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \end{aligned} \tag{4.60}$$

*for any $(x,t) \in Q_T$.*

**Definition 4.3** *If $F$ is a closed subset of $\mathbb{R}^N$, we define the $(2/q,q')$ integral capacitary potential $\mathcal{W}_F$ by*

$$\mathcal{W}_F(x,t) = t^{-1-N/2} \int_0^{D_F(x)} s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \quad \forall (x,t) \in Q_\infty, \tag{4.61}$$

*where $D_F(x) = \max\{|x-y| : y \in F\}$.*

An easy computation shows that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x,t) - t^{-(1+N/2)} \int_0^{\sqrt{ta_t}} s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} 0 \leq t^{-(1+N/2)} \int_0^{\sqrt{t(a_t+2)}} s^{N-2/(q-1)} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds - \mathcal{W}_F(x,t) \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \tag{4.63}$$

for some $C = C(N,q) > 0$. Furthermore

$$\mathcal{W}_F(x,t) = t^{-1/(q-1)} \int_0^{D_F(x)/\sqrt{t}} s^{N-2/(q-1)} e^{-s^2/4} C_{2/q,q'}\left(\frac{F}{s\sqrt{t}} \cap B_1(x)\right) s\,ds. \tag{4.64}$$

The following result gives a sufficient condition in order $\overline{u}_F$ has not a strong blow-up at some point $x$.

**Proposition 4.4** *Assume $q \geq q_c$ and $F$ is a closed subset of $\mathbb{R}^N$. If there exists $\gamma \in [0,\infty)$ such that*

$$\lim_{\tau \to 0} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma, \tag{4.65}$$

*then*

$$\lim_{t \to 0} t^{1/(q-1)} \overline{u}_F(x,t) = C\gamma, \tag{4.66}$$

*for some $C = C(N,q) > 0$.*

*Proof.* Clearly, condition (4.65 ) implies

$$\lim_{t\to 0} C_{2/q,q'}\left(\frac{F}{\sqrt{t}s}\cap B_1(x)\right) = \gamma$$

for any $s > 0$. Then (4.66 ) follows by Lebesgue's theorem. Notice also that the set of $\gamma$ is bounded from above by a constant depending on $N$ and $q$. □

In the next result we give a condition in order the solution remains bounded at some point $x$. The proof is similar to the previous one.

**Proposition 4.5** *Assume $q \geq q_c$ and $F$ is a closed subset of $\mathbb{R}^N$. If*

$$\limsup_{\tau\to 0} \tau^{-2/(q-1)} C_{2/q,q'}\left(\frac{F}{\tau}\cap B_1(x)\right) < \infty, \tag{4.67}$$

*then $\overline{u}_F(x,t)$ remains bounded when $t \to 0$.*

# A Appendix

The next estimate is crucial in the study of semilinear parabolic equations.

**Lemma A.1** *Let $a$ and $b$ be two real numbers, $a > 0$ and $\kappa > 0$. Then there exists a constant $C = C(a,b,\kappa) > 0$ such that for any $A > 0$, $B > \kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)}e^{-B^2/4x}dx \leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.1}$$

*Proof.* We first notice that

$$\max\{e^{-A^2/4(1-x)}e^{-B^2/4x} : 0 \leq x \leq 1\} = e^{-(A+B)^2/4}, \tag{A.2}$$

and it is achieved for $x_0 = B/(A+B)$. Set $\Phi(x) = (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)}e^{-B^2/4x}$, thus

$$\int_0^1 \Phi(x)dx = \int_0^{x_0} \Phi(x)dx + \int_{x_0}^1 \Phi(x)dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \tag{A.3}$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \tag{A.4}$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u}\left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}\right)$$

$$\int_0^{x_0} (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)-B^2/4x}dx = -\int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a}x(u)^{-b}e^{-u}x'(u)du$$

Putting $x' = x'(u)$ and differentiating (A.4 ),

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \Longrightarrow -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{4u + B^2 - A^2 - 8ux(u)}. \tag{A.5}$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{\sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}}, \tag{A.6}$$

and the factorization below holds

$$16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = v + (A+B)^2/4$ and obtain

$$x(u) = \frac{v + (AB + B^2)/2 - \sqrt{v(v+AB)}}{2\,(v + (A+B)^2/4)},$$

and

$$1 - x(u) = \frac{v + (A^2 + AB)/2 + \sqrt{v(v+AB)}}{2\,(v + (A+B)^2/4)}.$$

We introduce the relation $\approx$ linking two positive quantities depending on $A$ and $B$. It means that the two sided-inequalities up to multiplicative constants independent of $A$ and $B$. Therefore

$$\int_0^{x_0} \Phi(x)dx = 2^{a-b-4}e^{-(A+B)^2/4}\int_0^{\infty} \tilde{\Phi}(v)dv \quad \text{where}$$

$$\tilde{\Phi}(v) = \frac{\left(v + (AB+B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b}\left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a}}{(v + (A+B)^2/4)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.7}$$

*Case 1: $a \geq 1$, $b \geq 1$.* First

$$\frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\left(v + (A+B)^2\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \tag{A.8}$$

since $a + b - 2 \geq 0$ and $AB \geq \kappa$. Next

$$\left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a} \approx (v + A(A+B))^{1-a}. \tag{A.9}$$

Furthermore

$$\begin{aligned} v + (AB+B^2)/2 - \sqrt{v(v+AB)} &= B^2\frac{v + (A+B)^2/4}{v + B(A+B)/2 + \sqrt{v(v+AB)}} \\ &\approx B^2\frac{v + (A+B)^2}{v + B(A+B)}. \end{aligned} \tag{A.10}$$

Then

$$\left(v+(AB+B^2)/2-\sqrt{v(v+AB)}\right)^{1-b}\approx B^{2-2b}\left(\frac{v+B(A+B)}{v+(A+B)^2}\right)^{b-1} \tag{A.11}$$

It follows

$$\begin{aligned}\tilde{\Phi}(v)&\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{(v+B(A+B))^{b-1}}{\sqrt{v(v+\kappa)}}\\&\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{v^{b-1}+(B^2+AB)^{b-1}}{\sqrt{v(v+\kappa)}}\end{aligned} \tag{A.12}$$

where $C$ depends on $a$, $b$ and $\kappa$. The function $v\mapsto(v+(A+B)^2)/(v+A(A+B))$ is decreasing on $(0,\infty)$. If we set

$$C_1=\int_0^\infty\frac{v^{b-1}e^{-v}dv}{\sqrt{v(v+\kappa)}}\quad\text{and}\quad C_2=\int_0^\infty\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}}$$

then

$$C_1\leq K(B^2+AB)^{b-1}C_2$$

with $K=C_1\kappa^{1-b}/C_2$. Therefore

$$\int_0^{x_0}\Phi(x)dx\leq Ce^{-(A+B)^2/4}B^{1-b}A^{1-a}(A+B)^{a+b-2}. \tag{A.13}$$

The estimate of $J_{a,b}$ is obtained by exchanging $(A,a)$ with $(B,b)$ and replacing $x$ by $1-x$. *Mutadis mutandis*, this yields directely to the same expression as in A.13 and finally

$$\int_0^1\Phi(x)dx\leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.14}$$

*Case 2:* $a\geq 1$, $b<1$. Estimates (A.7 ), (A.8 ), (A.9 ), (A.10 ) and (A.11 ) are valid. Because $v\mapsto(v+B(A+B))^{b-1}$ is decreasing, (A.12 ) has to be replaced by

$$\tilde{\Phi}(v)\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{\left(AB+B^2\right)^{b-1}}{\sqrt{v(v+\kappa)}}. \tag{A.15}$$

This implies (A.13 ) directly. The estimate of $J_{a,b}$ is performed by the change of variable $x\mapsto 1-x$. If $x_1=1-x_0$ , there holds

$$J_{a,b}=\int_0^{x_1}x^{-a}(1-x)^{-b}e^{-A^2/4x}e^{-B^2/4(1-x)}dx=\int_0^{x_1}\Psi(x)dx.$$

Then

$$\int_0^{x_1}\Psi(x)dx=2^{b-a-4}e^{-(A+B)^2/4}\int_0^{x_1}\tilde{\Psi}(v)dv\quad\text{where}$$

$$\tilde{\Psi}(v)=\frac{\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a}\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b}}{(v+(A+B)^2/4)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.16}$$

Equivalence (A.8 ) is unchanged; (A.9 ) is replaced by

$$\left(v + (B^2 + AB)/2 + \sqrt{v(v + AB)}\right)^{1-b} \approx (v + B(A + B))^{1-b}, \tag{A.17}$$

(A.10 ) by

$$v + (AB + A^2)/2 - \sqrt{v(v + AB)} \approx A^2 \frac{v + (A + B)^2}{v + A(A + B)}, \tag{A.18}$$

and (A.11 ) by

$$\left(v + (AB + A^2)/2 - \sqrt{v(v + AB)}\right)^{1-a} \approx A^{2-2a} \left(\frac{v + A(A + B)}{v + (A + B)^2}\right)^{a-1}. \tag{A.19}$$

Because $a > 1$, (A.12 ) turns into

$$\begin{aligned}\tilde{\Psi}(v) &\leq CA^{2-2b}(v + (A + B)^2)^{b-1} \frac{(v + A^2 + AB)^{a-1}(v + B^2 + AB)^{1-b}}{\sqrt{v(v + \kappa)}} \\ &\leq Ce^{-(A+B)^2/4} A^{2-2b}(A + B)^{2b-2} \\ &\qquad \times \frac{v^{a-b} + (A^2 + AB)^{a-1}v^{1-b} + (B^2 + AB)^{1-b}v^{a-1} + A^{a-1}B^{1-b}(A + B)^{a-b}}{\sqrt{v(v + \kappa)}}.\end{aligned} \tag{A.20}$$

Because $AB \geq \kappa$, there exists a positive constant $C$, depending on $\kappa$, such that

$$\begin{aligned}&\int_0^\infty \frac{v^{a-b} + (A^2 + AB)^{a-1}v^{1-b} + (B^2 + AB)^{1-b}v^{a-1}}{\sqrt{v(v + \kappa)}} e^{-v} dv \\ &\qquad\qquad \leq CA^{a-1}B^{1-b}(A + B)^{a-b} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v + \kappa)}}.\end{aligned} \tag{A.21}$$

Combining (A.20 ) and (A.21 ) yields to

$$\int_0^{x_1} \Psi(x) dx \leq Ce^{-(A+B)^2/4} A^{1-a}B^{1-b}(A + B)^{a+b-2}. \tag{A.22}$$

This, again, implies that (A.1 ) holds.

*Case 3:* $\max\{a, b\} < 1$. Inequalities (A.7 )-(A.11 ) hold, but (A.12 ) has to be replaced by

$$\begin{aligned}\tilde{\Phi}(v) &\leq CB^{2-2b} \left(\frac{v + (A + B)^2}{v + A(A + B)}\right)^{a-1} \frac{\left(v + B^2 + AB\right)^{b-1}}{\sqrt{v(v + \kappa)}} \\ &\leq CB^{1-b}(A + B)^{2a+b-3} \frac{v^{1-a} + \left(A^2 + AB\right)^{1-a}}{\sqrt{v(v + \kappa)}}\end{aligned} \tag{A.23}$$

Noticing that

$$\int_0^\infty \frac{v^{1-a}e^{-v} dv}{\sqrt{v(v + \kappa)}} \leq C \left(A^2 + AB\right)^{1-a} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v + \kappa)}},$$

it follows that (A.13 ) holds. Finally (A.14 ) holds by exchanging $(A, a)$ and $(B, b)$. □

**Lemma A.2** . *Let $\alpha$, $\beta$, $\gamma$, $\delta$ be real numbers and $\ell$ an integer. We assume $\gamma > 1$, $\delta > 0$ and $\ell \geq 2$. Then there exists a positive constant $C$ such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell} p^{\alpha}(\sqrt{n} - \sqrt{p}\,)^{\beta} e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq C n^{\alpha-\beta/2} e^{-\delta n}. \tag{A.24}$$

*Proof.* The function $x \mapsto (\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1}, \infty)$. Furthermore there exists $C > 0$ depending on $\ell$, $\alpha$ and $\beta$ such that $p^{\alpha}(\sqrt{n} - \sqrt{p}\,)^{\beta} \leq C x^{\alpha}(\sqrt{n} - \sqrt{x+1}\,)^{\beta}$ for $x \in [p, p+1]$ If we denote by $p_0$ the smallest integer larger than $(\gamma - 1)^{-1}$, we derive

$$\begin{aligned} S = \sum_{p=1}^{n-\ell} p^{\alpha}(\sqrt{n} - \sqrt{p}\,)^{\beta} e^{-(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2/4} &= \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^{\alpha}(\sqrt{n} - \sqrt{p}\,)^{\beta} e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ \leq \sum_{p=1}^{p_0-1} p^{\alpha}(\sqrt{n} - \sqrt{p}\,)^{\beta} e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} & \\ &+ C \int_{p_0}^{n+1-\ell} x^{\alpha}(\sqrt{n} - \sqrt{x}\,)^{\beta} e^{-\delta(\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2} dx, \end{aligned}$$

(notice that $\sqrt{n} - \sqrt{x} \approx \sqrt{n} - \sqrt{x+1}$ for $x \leq n - \ell$). Clearly

$$\sum_{p=1}^{p_0-1} p^{\alpha}(\sqrt{n} - \sqrt{p}\,)^{\beta} e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq C_0 n^{\alpha}(\sqrt{n} - \sqrt{n-\ell}\,)^{\beta} e^{-\delta n} \tag{A.25}$$

for some $C_0$ independent of $n$. We set $y = y(x) = \sqrt{x+1} - \sqrt{x}/\sqrt{\gamma}$. Obviously

$$y'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{\gamma}\sqrt{x}}\right) \quad \forall x \geq p_0,$$

and their exists $\epsilon = \epsilon(\delta, \gamma) > 0$ such that $\sqrt{2}\sqrt{x} \geq y(x) \geq \epsilon\sqrt{x}$ and $y'(x) \geq \epsilon/\sqrt{x}$. Furthermore

$$\sqrt{x} = \frac{\sqrt{\gamma}\left(y + \sqrt{\gamma y^2 + 1 - \gamma}\right)}{\gamma - 1},$$

$$\begin{aligned} \sqrt{n} - \sqrt{x} &= \frac{\sqrt{n}(\gamma - 1) - \sqrt{\gamma}y - \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}}{\gamma - 1} \\ &= \frac{n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}(\gamma - 1) - \sqrt{\gamma}y + \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}} \\ &\approx \frac{n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}} \end{aligned}$$

since $y(x) \leq \sqrt{n}$. Furthermore

$$\begin{aligned} n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2 &= \gamma(\sqrt{n+1} + \sqrt{n}/\sqrt{\gamma} + y)(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y) \\ &\approx \sqrt{n}(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y), \end{aligned}$$

because $y$ ranges between $\sqrt{n+2-\ell} - \sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$ and $\sqrt{p_0+1} - \sqrt{p_0}\sqrt{\gamma}$. Thus

$$(\sqrt{n} - \sqrt{x})^\beta \approx \left(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y\right)^\beta .$$

This implies

$$\begin{aligned}\int_{p_0}^{n+1-\ell} & x^\alpha (\sqrt{n} - \sqrt{x})^\beta e^{-\delta(\sqrt{x}+\gamma(\sqrt{n}-\sqrt{x+1}))^2} dx \\ & \leq C \int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1} \left(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y\right)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2} dy \\ & \leq C n^{\alpha+\beta/2+1} \int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz.\end{aligned} \tag{A.26}$$

Moreover

$$\begin{aligned} 1 - \frac{y(p_0)}{\sqrt{n}} &= 1 - \frac{1}{\sqrt{n}} \left( \sqrt{p_0+1} - \frac{\sqrt{p_0}}{\sqrt{\gamma}} \right), \\ 1 - \frac{y(n-\ell+1)}{\sqrt{n}} &= 1 - \frac{\sqrt{n-\ell+2}}{\sqrt{n}} + \frac{\sqrt{n-\ell+1}}{\sqrt{n\gamma}} \\ &= \frac{1}{\sqrt{\gamma}} \left( 1 + \frac{\sqrt{\gamma}\,(\ell-2) - \ell + 1}{2n} + \frac{\sqrt{\gamma}\,(\ell-2)^2 - (\ell-1)^2}{8n^2} \right) + O(n^{-3}). \end{aligned} \tag{A.27}$$

Let $\theta$ fixed such that $1 - \dfrac{y(n-\ell+1)}{\sqrt{n}} < \theta < 1 - \dfrac{y(p_0)}{\sqrt{n}}$ for any $n > p_0$. Then

$$\begin{aligned} \int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz &\leq C_\theta \int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} e^{-\gamma\delta n z^2} dz \\ &\leq C_\theta\, e^{-\gamma\delta n\theta^2} \int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} dz \\ &\leq C\, e^{-\gamma\delta n\theta^2} \max\{1, n^{-\alpha-1/2}\}. \end{aligned}$$

Because $\gamma\theta^2 > 1$ we derive

$$\int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz \leq C n^{-\beta} e^{-\delta n}, \tag{A.28}$$

for some constant $C > 0$. On the other hand

$$\begin{aligned}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} & (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz \\ & \leq C'_\theta \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz.\end{aligned}$$

The minimum of $z \mapsto (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta$ is achieved at $1 - y(n+1-\ell)$ with value

$$\frac{\sqrt{\gamma}(\ell+1) + 1 - \ell}{2n\sqrt{\gamma}} + O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta+((\ell-2)\sqrt{\gamma}+1-\ell)/2}(1+\circ(1)) = C_\gamma e^{-n\delta}(1+\circ(1)).$$

We denote

$$z_{\gamma,n} = 1+1/\sqrt{\gamma}-\sqrt{1+1/n} \quad \text{and} \quad I_\beta = \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} dz.$$

Since $1-y(n+1-\ell) \geq 1/\sqrt{2\gamma}$ for $n$ large enough,

$$\begin{aligned}
I_\beta &\leq \sqrt{2\gamma}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z-z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2} dz \\
&\leq \frac{-\sqrt{2\gamma}}{2n\gamma\delta}\left[(z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2}\right]_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z-z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2} dz
\end{aligned}$$

But $1-y(n+1-\ell)/\sqrt{n}-z_{\gamma,n} = (\ell-1)(1-1/\sqrt{\gamma})/2n$, therefore

$$I_\beta \leq C_1 n^{-\beta-1}e^{-\delta n} + \beta C_1' n^{-1} I_{\beta-1}. \tag{A.29}$$

If $\beta \leq 0$ , we derive

$$I_\beta \leq C_1 n^{-\beta-1}e^{-\delta n},$$

which inequality, combined with (A.26 )and (A.28 ), yields to(A.24 ). If $\beta > 0$, we iterate and get

$$I_\beta \leq C_1 n^{-\beta-1}e^{-\delta n} + C_1' n^{-1}(C_1 n^{-\beta}e^{-\delta n} + (\beta-1)C_1' n^{-1} I_{\beta-2})$$

If $\beta - 1 \leq 0$ we derive

$$I_\beta \leq C_1 n^{-\beta-1}e^{-\delta n} + C_1 C_1' n^{-1-\beta}e^{-\delta n} = C_2 n^{-\beta-1}e^{-\delta n},$$

which again yields to (A.24 ). If $\beta - 1 > 0$, we continue up we find a positive integer $k$ such that $\beta - k \leq 0$, which again yields to

$$I_\beta \leq C_k n^{-\beta-1}e^{-\delta n}$$

and to (A.24 ). □

The next estimate is fundamental in deriving the $N$-dimensional estimate.

**Lemma A.3** *For any integer $N \geq 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m\cos\theta}\sin^{N-2}\theta\, d\theta \leq c_N \frac{e^m}{(1+m)^{(N-1)/2}} \quad \forall m > 0. \tag{A.30}$$

*Proof.* Put $\mathcal{I}_N(m) = \int_0^\pi e^{m\cos\theta}\sin^{N-2}\theta\, d\theta$. Then $\mathcal{I}_2'(m) = \int_0^\pi e^{m\cos\theta}\cos\theta\, d\theta$ and

$$\begin{aligned}\mathcal{I}_2''(m) &= \int_0^\pi e^{m\cos\theta}\cos^2\theta\, d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m\cos\theta}\sin^2\theta\, d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\, d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m}\mathcal{I}_2'(m).\end{aligned}$$

Thus $\mathcal{I}_2$ satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}_2'(0) = 0$, $\pi^{-1}\mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by $I_0$) the asymptotic behaviour of which is well known, thus (A.30 ) holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m\cos\theta}\sin\theta\, d\theta = \left[\frac{-e^{m\cos\theta}}{m}\right]_0^\pi = \frac{2\sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m}\frac{d}{d\theta}(e^{m\cos\theta})\sin^{N-3}\theta\, d\theta = \frac{N-3}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\sin^{N-4}\theta\, d\theta. \tag{A.31}$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\, d\theta = \mathcal{I}_2'(m),$$

and, again (A.30 ) holds since $I_0'(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2}\left[e^{m\cos\theta}\cos\theta\sin^{N-5}\theta\right]_0^\pi + \frac{N-3}{m^2}\int_0^\pi e^{m\cos\theta}\frac{d}{d\theta}\left(\cos\theta\sin^{N-5}\theta\right)d\theta.$$

Differentiating $\cos\theta\sin^{N-5}\theta$ and using (A.31 ), we obtain

$$\mathcal{I}_5(m) = \frac{4\sinh m}{m^2} - \frac{4\sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2}\left(\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)\right), \tag{A.32}$$

for $N \geq 6$. Since the estimate (A.30 ) for $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ has already been obtained, a straigthforward induction yields to the general result. □

*Remark.* Although it does not has any importance for our use, it must be noticed that $\mathcal{I}_N$ can be expressed either with hyperbolic functions if $N$ is odd, or with Bessel functions if $N$ is even.

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